Circular motion, Chapter 3

When we describe circular motion it is most convenient to use polar coordinates and radial as well as tangential unit vectors. We affix these unit vectors to the origin of a coordinate system which travels with the moving particle. One unit vector $\vec{u}_r$ always points in the direction of the radial vector $\vec{r}$, the other unit vector $\vec{u}_\theta$ is perpendicular to $\vec{u}_r$ and points in the tangential direction parallel to the tangential component of the velocity $\vec{u}_r$ of the moving particle.

To get the velocity in polar coordinates we form the derivative of the polar coordinates:

(1.1)

Polar coordinates in the plane. As the angular position changes, the angle θ is a function of time: $\theta(t)=\omega t$:

$$x(t) = r(t)\cos\left[\theta(t)\right]; x = r\cos\omega t; y = r\sin\omega t;$$

In uniform circular motion there is no tangential acceleration, centripetal acceleration only. We see soon what that means:

(1.2) $$\vec{r} = \left\langle r\cos\omega t, r\sin\omega t\right\rangle = r\underbrace{\left\langle \cos\omega t, \sin\omega t\right\rangle}_{\vec{u}_r} = r\vec{u}_r$$

It is easy to see that the vector $\underbrace{\left\langle \cos\omega t, \sin\omega t\right\rangle}_{\vec{u}_r} = \vec{u}_r$ is a unit vector. The magnitude of the radius vector is given by:

(1.3) $$r = \sqrt{x^2 + y^2}; \theta = \tan^{-1}\frac{y}{x}$$

If we form the derivative of the components we can find the **magnitude** of the velocity in angular coordinates:

(1.4) $$\vec{\mathrm{v}} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left\langle r\cos\omega t, r\sin\omega t\right\rangle = \left\langle -r\omega\sin\omega t, r\omega\cos\omega t\right\rangle =$$
$$\dot{\vec{r}} = \vec{\mathrm{v}} = r\omega\left\langle -\sin\omega t, \cos\omega t\right\rangle = r\omega\vec{u}_\theta$$

We proceed in the same way to find the expression for the acceleration in uniform circular motion. All we need to do is to apply the rules of derivation to (1.4).

(1.5) $$\vec{\mathrm{a}} = \frac{d}{dt}\vec{\mathrm{v}} = = \frac{d}{dt}\left[r\omega\left\langle -\sin\omega t, \cos\omega t\right\rangle\right] = r\omega\left\langle -\omega\cos\omega t, -\omega\sin\omega t\right\rangle =$$
$$= -r\omega^2\left\langle \cos\omega t, \sin\omega t\right\rangle = -r\omega^2\vec{u}_r = -\frac{\mathrm{v}^2}{\mathrm{r}}\vec{u}_r$$

The magnitude of this vector is $r\omega^2 = \frac{\mathrm{v}^2}{\mathrm{r}}$

The direction of this vector is opposite to the direction of the radius vector and thus always points to the center of the circular motion; this acceleration is therefore called centripetal acceleration.

These equations taken together show also that the derivative of the unit vector $\vec{u}_r$ is equal to $\omega\cdot\vec{u}_\theta$ where $\vec{u}_\theta$ is the tangential unit vector, perpendicular to $\vec{u}_r$. Furthermore, the derivative of the tangential unit vector is equal $-\omega\cdot\vec{u}_r$.

(1.6)
$$\vec{u}_r = \langle \cos\omega t, \sin\omega t\rangle; \vec{u}_\theta = \langle -\sin\omega t, \cos\omega t\rangle$$
$$\frac{d}{dt}\vec{u}_r = \omega\vec{u}_\theta; \frac{d}{dt}\vec{u}_\theta = -\omega\vec{u}_r$$

If we allow for circular motion with tangential acceleration the angular velocity ω(t) changes with time.

(1.7)
$$\frac{d}{dt}\omega = \dot{\omega} = \alpha \text{ angular acceleration in radians/s}^2$$

When we take the derivative of the velocity vector (1.4) we get the resultant acceleration

(1.8)
$$\vec{a} = \underbrace{r\alpha\vec{u}_\theta}_{\text{tangential acceleration}} \quad \underbrace{-r\omega^2\vec{u}_r}_{\text{centripetal acceleration}}$$

(1.9)
$$\boxed{\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(r\omega\vec{u}_\theta\right) = r\frac{d\omega}{dt}\vec{u}_\theta - r\omega^2\vec{u}_r = r\alpha\vec{u}_\theta - r\omega^2\vec{u}_r}$$

The first term corresponds to tangential acceleration, the second to centripetal acceleration.

If we consider a more general case in which the length of the radius vector r is not constant, we have to apply the product rule of derivatives to both r and the unit vector:

(1.10) $$\frac{dx}{dt} = \dot{x} = \dot{r}\cos\omega t - r\omega\sin\omega t; \frac{dy}{dt} = \dot{y} = \dot{r}\sin\omega t + r\omega\cos\omega t$$

(1.11)
$$v=\sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{\left(\dot{r}\cos\omega t - r\omega\sin\omega t\right)^2 + \left(\dot{r}\sin\omega t + r\omega\cos\omega t\right)^2} =$$
$$\sqrt{\dot{r}^2\cos^2\omega t + r^2\omega^2\sin^2\omega t - 2\dot{r}\cos\omega t\cdot r\omega\sin\omega t + \dot{r}^2\sin^2\omega t + r^2\omega^2\cos^2\omega t + 2\dot{r}\cos\omega t\cdot r\omega\sin\omega t} =$$
$$\sqrt{\left(\dot{r}^2 + r^2\omega^2\right)\left(\cos^2\omega t + \sin^2\omega t\right)} = \sqrt{\left(\dot{r}^2 + r^2\omega^2\right)}$$

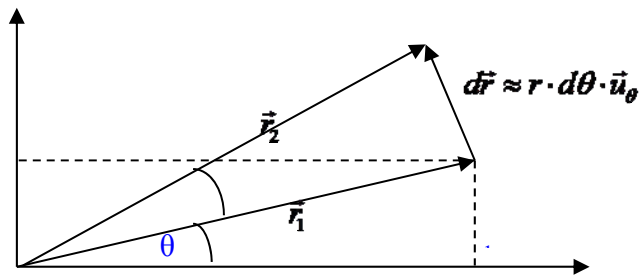
This last expression contains a change of the radius dr/dt and a change of the angle dθ/dt which we call ω. The expression r·dθ is the change of the arc length with the increase in the angle. From the drawing below one can see that the velocity vector can be expressed in terms of a velocity in the radial direction plus a velocity in the tangential direction:

(1.12) $$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\vec{u}_r + r\frac{d\vec{u}_r}{dt} = \dot{r}\vec{u}_r + r\omega\vec{u}_\theta$$

where we have used the product rule for derivations.

We can get this result graphically.

We need to find the expression for the derivative of the unit-vector $\vec{u}_r$, which is in the case of a plane motion (circle in a plane) equal to $\omega \cdot \vec{u}_\theta$. Let us see why that is so:



(1.13) $$\frac{d\vec{r}}{dt} = \vec{v} = \lim_{\Delta t \to 0}\frac{\vec{r}_2 - \vec{r}_1}{\Delta t} = \lim_{\Delta t \to 0}\frac{\Delta\vec{r}}{\Delta t}$$

Its direction is tangential, given by the product of the tangential unit vector $\vec{u}_\theta$ and the magnitude $v = r\frac{d\theta}{dt} = r\dot{\theta} = r\omega$

In the last formula we approximate Δr by the arc-length spanned by the angle dθ.

Therefore, velocity for uniform circular motion is given by a tangential vector with the magnitude of rω. If the object moves also in the original radial direction, we have to add the vector in the radial direction:

(1.14) $$\boxed{\frac{dr}{dt}\vec{u}_r = \dot{r}\vec{u}_r}$$

For a pure circular motion the term dr/dt is 0 and we get the tangential velocity alone:

(1.15) $$\vec{v}_\theta = \frac{d\vec{r}}{dt} = \omega r\vec{u}_\theta$$

Therefore equation (1.12) for the velocity vector becomes in general, i.e. for a motion in the tangential and the radial direction; in polar coordinates):

(1.16) $$\vec{v} = r\frac{d\theta}{dt}\vec{u}_\theta + \frac{dr}{dt}\vec{u}_r = r\dot{\theta}\vec{u}_\theta + \dot{r}\vec{u}_r = r\omega\vec{u}_\theta + \dot{r}\vec{u}_r$$

All our work with polar coordinates boils down to knowing the derivatives of the unit vectors in the radial and tangential directions. This is why I spend so much time on them. They remain the same even if we consider a situation with changing angular velocity.

Memorize:

$$\boxed{\begin{array}{l} \frac{d}{dt}\vec{u}_r = \omega\vec{u}_\theta \text{ and } \frac{d}{dt}\vec{u}_\theta = -\omega\vec{u}_r \\ \vec{r} = r\vec{u}_r \end{array}}$$