Lect 9, Universal Gravitation

(include angular momentum)

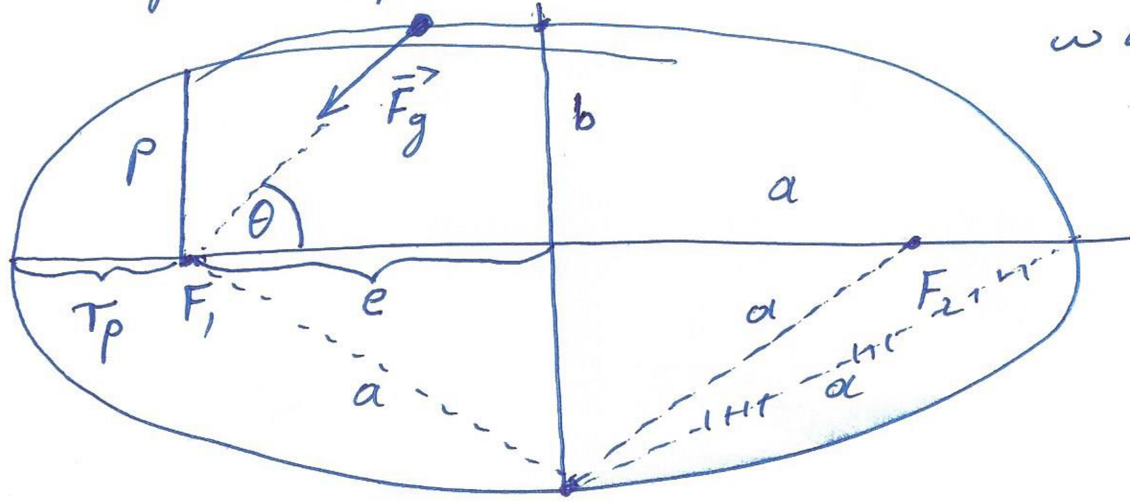
$$\vec{F}_g = -\frac{GmM}{r^2}\vec{u}_r$$

Kepler's 1st law.

All planets and other returning bodies like satellites and comets move around the sun in elliptical paths.

$\theta = \omega \cdot t$

ω is not constant



$$r = \frac{p}{1 - \varepsilon \cos\theta}$$ polar representation of an ellipse

$r_p + e = a$; $\varepsilon = \frac{e}{a}$ = eccentricity of the ellipse.

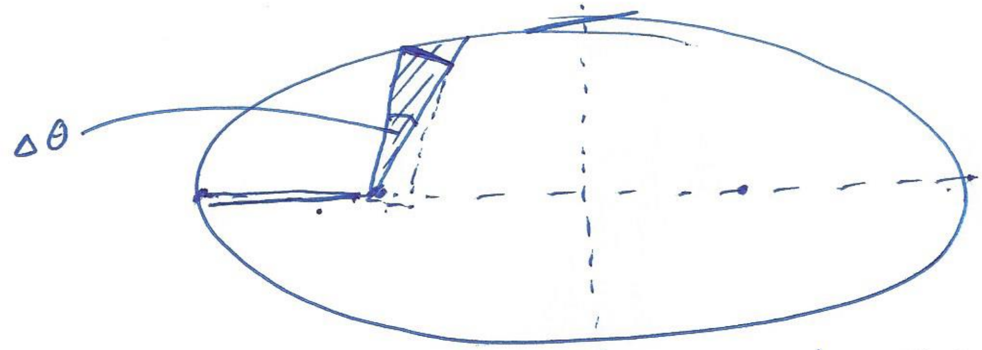
$r_p = 1.300\,AU$; $a = 28.23\,AU$; $e = a - r_p = 28.23 - 1.300 = 26.93\,AU$

$$\varepsilon = \frac{26.93}{28.23} = \underline{\underline{0.9539}}$$

$$b = a\sqrt{1-\varepsilon^2}$$
$$= 28.23\sqrt{1-0.9539^2} = \underline{\underline{8.468\,AU}}$$

Kepler's 3rd law:

area = $\pi \cdot a \cdot b$



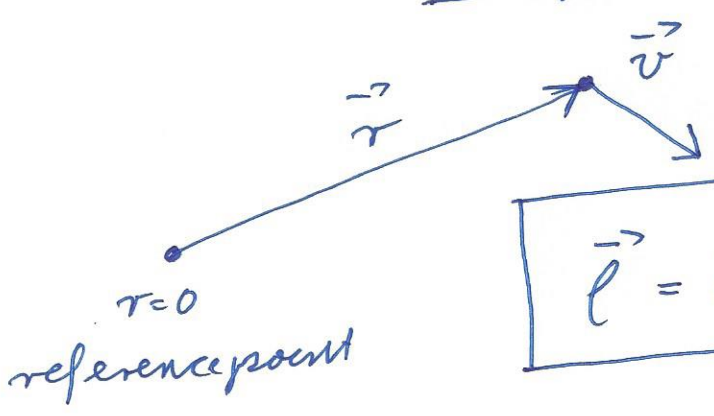
ΔA = small area swept over by r

$$\frac{\Delta A}{\Delta t} = \frac{r \cdot r \cdot \Delta\theta}{2\,\Delta t} = \frac{r^2}{2} \cdot \frac{\Delta\theta}{\Delta t} = \frac{r^2\,\omega}{2}$$

$r^2 \cdot \omega$ is a constant.

$r(t)$, $\omega(t)$, $v(t)$ all change as the planet moves around the sun.

Angular momentum of a point



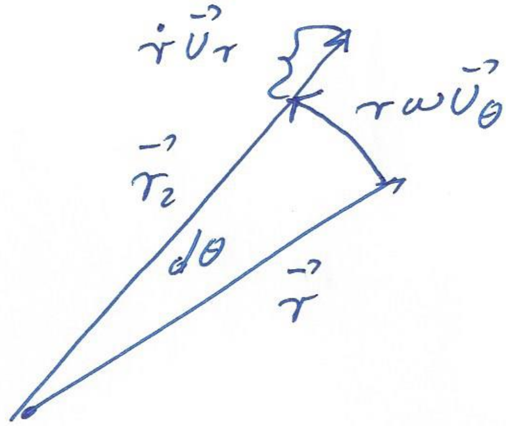
$$\boxed{\vec{\ell} = \vec{r} \times m\vec{v}}$$

For gravitational forces $\vec{\ell} = \overrightarrow{\text{constant}}$

Show that $\dfrac{d\vec{\ell}}{dt} = \vec{0}$

$\vec{l} = \vec{r} \times m\vec{v};$ $\quad \vec{r} = r\vec{U}_r$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\vec{U}_r + r\omega\vec{U}_\theta$$



$$\vec{l} = \vec{r} \times m(\dot{r}\vec{U}_r + r\omega\vec{U}_\theta)$$

$$\vec{r} \times \vec{U}_r = \vec{0}$$

$$\vec{l} = \vec{r} \times m \cdot r\omega\vec{U}_\theta$$
$$= r\vec{U}_r \times m \cdot r\omega\vec{U}_\theta$$
$$= r^2\omega \cdot m \underbrace{\vec{U}_r \times \vec{U}_\theta}_{\vec{U}_z}$$

$$\boxed{\vec{l} = r^2\omega m\vec{U}_z}$$ $\quad l = r^2\omega m$

$r^2\omega$ = constant = Kepler's third law

Prove that $\frac{d\vec{\ell}}{dt} = \vec{0}$

$$\vec{\ell} = \vec{r} \times m\vec{v}$$

$$\frac{d\vec{\ell}}{dt} = \frac{d\vec{r}}{dt} \times m\vec{v} + \vec{r} \times m\frac{d\vec{v}}{dt}$$

$$= \underbrace{\cancel{\vec{v} \times m\vec{v}}}_{\vec{0}} + \vec{r} \times \underbrace{m\vec{a}}_{\vec{F}_g}$$

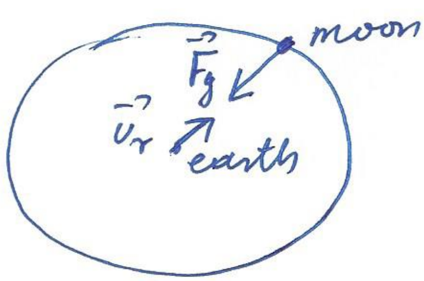
$$+ \vec{r} \times (-)\frac{mMG}{r^2}\vec{u}_r$$

$$= \vec{0}$$

Calculate the distance between moon and earth, knowing the mass of the earth

$$m_e = 5.98 \cdot 10^{24}\,kg$$

Period = 27 days!



$$-\frac{m_e \cdot \cancel{m_m} \cdot G}{a^2} = -\cancel{m_m}\omega^2 a$$

$$\vec{F}_g = m\vec{a}$$

$$\frac{m_e \cdot G}{a^3} = \frac{4\pi^2}{T^2}$$

$$\frac{m_e \cdot G}{4\pi^2} = \frac{a^3}{T^2} = \frac{5.98 \cdot 10^{24} \cdot 6.673 \cdot 10^{-11}}{4\pi^2}$$

$$= \underline{\underline{1.01 \cdot 10^{13}}}$$

$$a^3 = 1.01 \cdot 10^{13} \cdot T^2$$
$$= 1.01 \cdot 10^{13} \cdot (27 \cdot 24 \cdot 3600)^2$$
$$= 5.5 \cdot 10^{25}$$

$$a = 3.8 \cdot 10^8 \, m$$

speed of the moon? $v = \omega \cdot r = \frac{2\pi r}{T} = \frac{2\pi a}{T}$

$$= \frac{2\pi \cdot 3.8 \cdot 10^8}{27 \text{days}} = 1.02 \cdot 10^3 \frac{m}{s}$$

acceleration $= \frac{v^2}{r} = \omega^2 \cdot r$

Speed of a satellite around the earth at a distance of 5000 km above the surface.

$r_e = 5.67 \cdot 10^6 m = 5.67 \cdot 10^3 km$

$a = \cancel{10.67 \cdot 10^6 m} = 10.67 \cdot 10^6 m$

~~5000 km + 5.6700~~

$\frac{a^3}{T^2} = \text{const.}$ $\quad T^2 = \frac{a^3}{\text{constant}}$

$$T^2 = \frac{(10.67 \cdot 10^6)^3}{1.01 \cdot 10^{13}}$$

$v = \frac{2\pi r}{T}$ $\quad T = 1.097 \cdot 10^4 s = 3.046 \text{ hours.}$

Speed of a tree top satellite

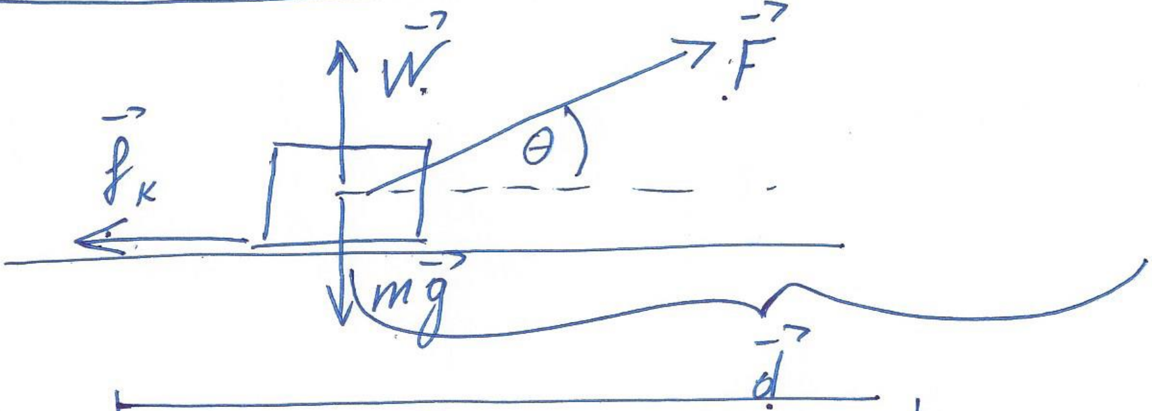
$$\frac{\cancel{m} v^2}{r} = \frac{\cancel{m} M G}{r^2}$$

$$v^2 = \underbrace{\frac{MG}{r^2}}_{g} \cdot r$$

$$v = \sqrt{g \cdot r} = \sqrt{9.8 \cdot 6.37 \cdot 10^6}$$

$$= 7901 \frac{m}{s}$$

Chapter 7: Work and energy

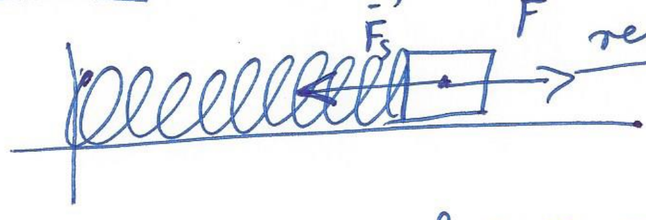


$$\boxed{\begin{aligned} W = \vec{F} \cdot \vec{d} &= F_x\, x + F_y \cdot y \\ &= F \cdot d \cdot \cos\theta \end{aligned}}$$

$$W_{f_K} = \vec{f}_K \cdot \vec{d} = -f_K \cdot d$$

$$W_F = \vec{F} \cdot \vec{d} = F \cos\theta \cdot d$$

Spring force: $\vec{F}_s = -k\,\underline{x}\,\vec{i}$

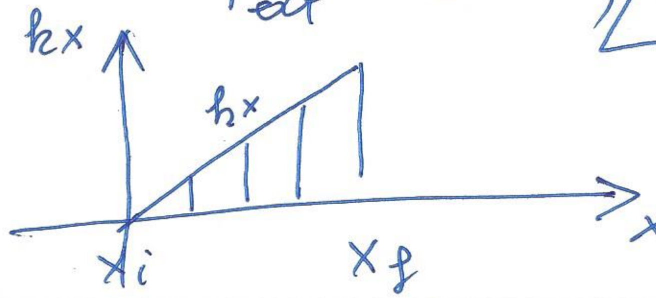


restoring force.

k = spring constant.

$F_{ext} = kx$

$$\sum_i \Delta W = \sum k x_i \Delta x_i \rightarrow \int_{x_i}^{x_f} kx\, dx$$



$$W = \int_{x_i}^{x_f} F(x)\,dx$$