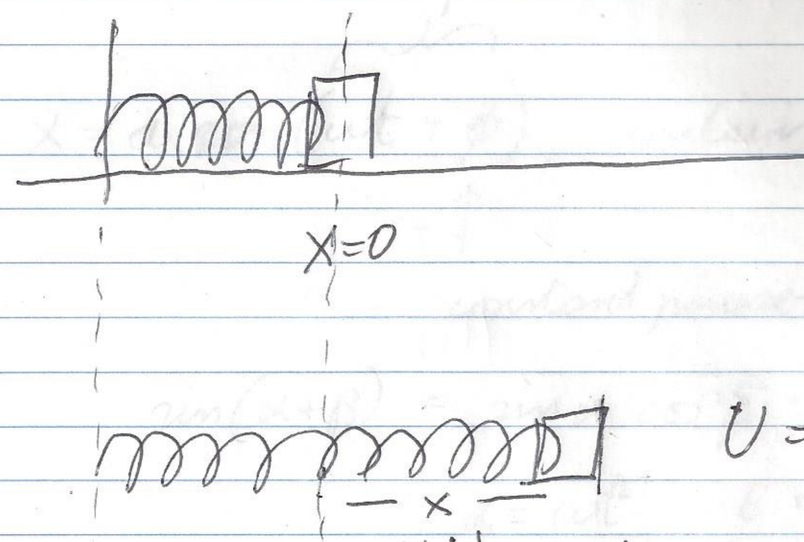
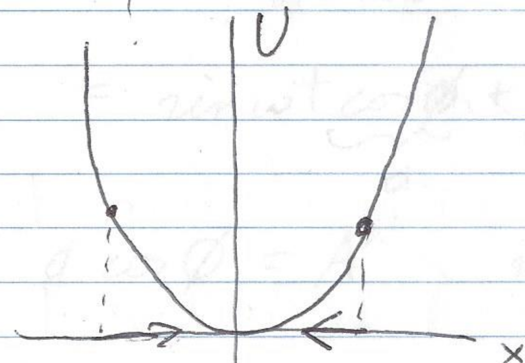
# Lecture 17 Oscillations

$F = -kx$ force of a spring




frictionless surface

$U = \frac{1}{2} k x^2$




$F = -\frac{d}{dx} U$

restoring force

$F = ma = -kx$

$m\ddot{x} + kx = 0$ $\quad$ $x(t)$ whose second derivative is proportional to $-x$

$\ddot{x} + \frac{k}{m} x = 0$

$x = A \cos \omega t$ or $x = B \sin \omega t$

$\dot{x} = -\omega A \sin \omega t$

$\ddot{x} = -\omega^2 A \cos \omega t$; $\quad \omega^2 = \frac{k}{m}$

SHM = simple harmonic motion.

$$\omega = \omega_0 = \sqrt{\frac{k}{m}}$$

$$x = a \cdot \sin(\overbrace{\omega t + \phi}^{\text{phase}});$$ contains all solutions

$\phi$: constant phaseshift

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$\alpha = \omega t \qquad \beta = \phi$$

$$= \sin\omega t \underbrace{\cos\phi}_{a} + \underbrace{\sin\phi}_{a} \cos\omega t$$

$$a\cos\phi = A \qquad \sin\phi \cdot a = B$$

$$\left.\begin{aligned} x &= A\sin\omega t + B\cos\omega t \\ &= a\sin(\omega t + \phi) \end{aligned}\right\}$$ general solution of the d.e.

We get a unique solution by imposing initial conditions $x(0) =$ $\qquad v(0) =$

The standard correspond to a fully expanded spring at $t=0$ with $v=0$

$$x(t=0) = A \qquad v(t=0) = 0$$

Total energy is conserved $E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$

$= \text{constant}.$

$x(t=0) = x_0$

$v(t=0) = 0$

We start with the general solution and impose these standard initial conditions!

$x = A\sin\omega t + B\cos\omega t$

$x(t=0) = x_0 = A\sin 0 + B\cos 0 = B$

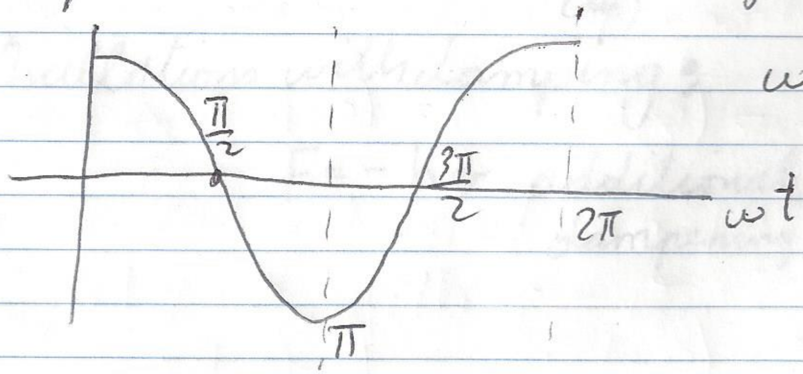
$\dot{x} = \omega A\cos\omega t + (-1)\omega B\sin\omega t$

$t=0 \quad 0 = \omega A \cdot \cancel{\cos 0} - B\omega\sin 0$

$A = 0$

$x = B\cos\omega t = x_0\cos\omega t$

Unique standard solution of the d.e.




$\omega t = 2\pi$

$T = \frac{2\pi}{\omega}$

Energy:

$$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

$$= \frac{1}{2}k(x_0 \cos\omega t)^2 + \frac{1}{2}m(-x_0\omega \sin\omega t)^2$$

$$E = \frac{1}{2}kx_0^2 \cos^2\omega t + \frac{1}{2}m\, x_0^2\omega^2 \sin^2\omega t$$

$$\omega^2 = \frac{k}{m}$$

$$= \frac{1}{2}kx_0^2 \cos^2\omega t + \frac{1}{2}\cancel{m}\, x_0^2 \frac{k}{\cancel{m}} \sin^2\omega t$$

$$= \frac{1}{2}kx_0^2 \underbrace{(\cos^2\omega t + \sin^2\omega t)}_{1} = \frac{1}{2}kx_0^2$$

$$\frac{dE}{dt} = \frac{1}{\cancel{2}}k\cdot \cancel{2}x \underbrace{\frac{dx}{dt}}_{v} + \frac{1}{\cancel{2}}m\cdot\cancel{2}v\underbrace{\frac{dv}{dt}}_{a}$$

$$= kxv + mva = v\underbrace{(kx + ma)}_{0}$$ ✓

Oscillations with damping:
(en)

$F = -bv$ additional force of dampening.

$F_s = -kx \quad F_d = -bv$

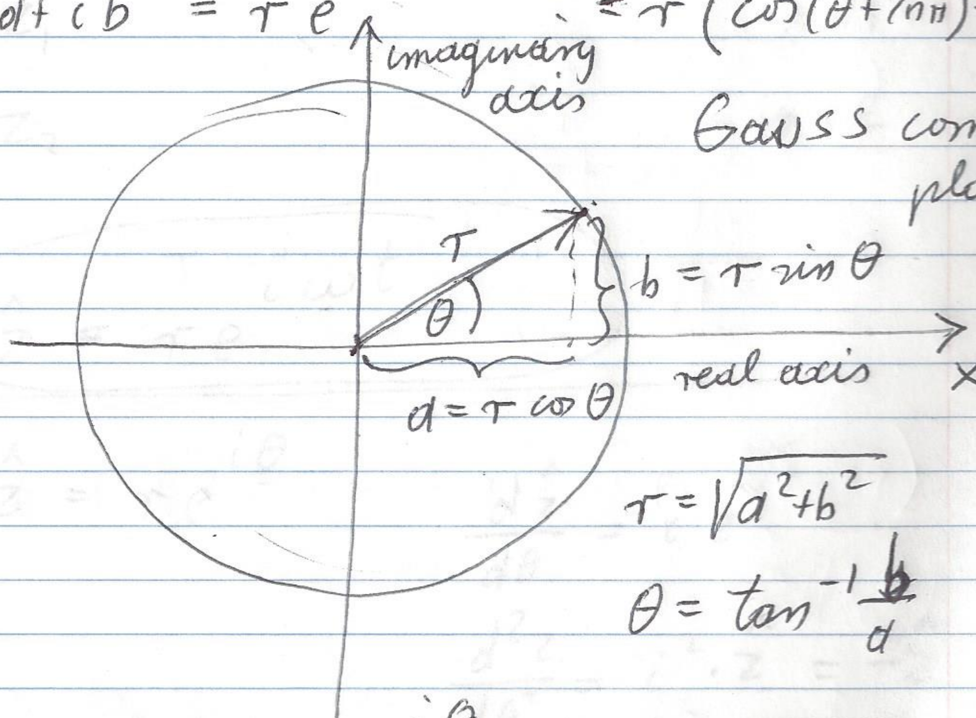
$$ma = -kx - bv$$

$$ma + bv + kx = 0$$

$$\boxed{\ddot{x} + \frac{b}{m}\dot{x} + \omega_0^2 \cdot x = 0} \qquad \omega_0^2 = \frac{k}{m}$$

Complex numbers:

$$a + ib = re^{i(\theta + 2n\pi)} = r\left(\cos(\theta + 2n\pi) + i\sin(\,)\right)$$




Gauss complex plane

$$r = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\frac{b}{a}$$

$$3 + 4i = re^{i\theta} \qquad r = \sqrt{9+16} = 5$$

$$= 5e^{i(0.927 + 2n\pi)} \qquad \theta = \tan^{-1}\frac{4}{3} \qquad \theta = 0.927$$

~~d+i~~

$\hat{z}$ = notation for complex numbers

$$\hat{z} = x + iy$$

$$z^3 = 3 + 4i = 5e^{i(0.927 + 2n\pi)}$$

$$z_0 = 5^{\frac{1}{3}} e^{\frac{i(0.927 + 2n\pi)}{3}} \qquad n = 0$$

$$= 5^{\frac{1}{3}} e^{\frac{i\,0.927}{3}}$$

$z_1$ $\qquad n = 1$

$z_2$ $\qquad n = 2$

$$\hat{z} = re^{i\omega t}$$

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$$\hat{z} = re^{i\theta}$$

$$\frac{d\hat{z}}{d\theta} = i\hat{z}$$

$$\frac{d^2 z}{d\theta^2} = i^2 \cdot \hat{z} = -\hat{z}$$

$$\hat{z} = re^{i\omega t}$$

$$\mathring{\hat{z}} = \frac{d\hat{z}}{dt} = i\omega\hat{z}$$

$$\ddot{\hat{z}} = \frac{d^2\hat{z}}{dt^2} = (i\omega)^2\hat{z} = -\omega^2 \cdot \hat{z}$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \omega_0^2 x = 0$$

$$\hat{z} = \text{trial solution} = A \cdot e^{i\alpha t}$$

$\hat{z} \to x$ $\quad\quad$ $\dot{x} = i\alpha x$ $\quad\quad$ x is now complex

$\ddot{x} = -\alpha^2 x$

$$-\alpha^2 x + \frac{b}{m} i\alpha x + \omega_0^2 x = 0$$

$$x \cdot \underbrace{\left(-\alpha^2 + i\frac{b}{m}\alpha + \omega_0^2\right)}_{=0} = 0$$

$$\alpha^2 - i\frac{b}{m}\alpha - \omega_0^2 = 0$$

$$\alpha = \frac{+i\frac{b}{m} \pm \sqrt{-\frac{b^2}{m^2} + 4\omega_0^2}}{2}$$

$$= i\frac{b}{2m} \pm \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

$$\alpha = +\frac{ib}{2m} \pm \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

$$X = Ae^{i\alpha t}$$

$$X = Ae^{it\cdot\alpha} = Ae^{it\left(+\frac{ib}{2m} \pm \underbrace{\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}}_{\omega_1}\right)}$$

$$= Ae^{-\frac{b}{2m}t}\, e^{\pm i\omega_1 t}$$

Euler formula: $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

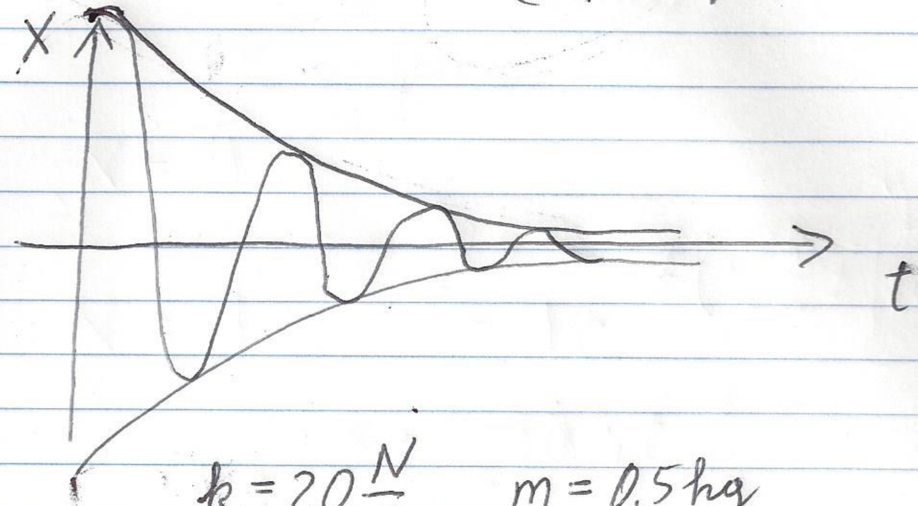
$$1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots$$

$$= \underbrace{1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!}}_{\cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\right)}_{\sin\theta}$$

$$\cos\theta \quad + i\sin\theta$$

$$X = Ae^{-\frac{b}{2m}t}\left(\cos\omega_1 t \pm i\sin\omega_1 t\right)$$

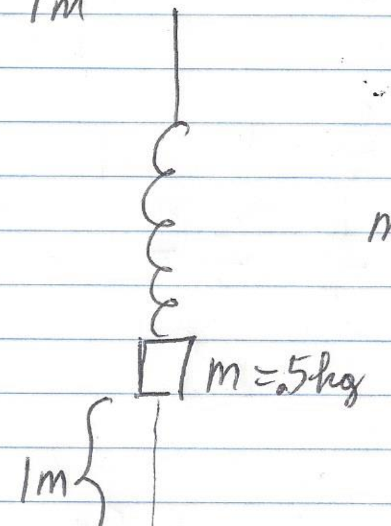
$$X = Ae^{-\frac{b}{2m}t}\cos(\omega_1 t + \phi)$$




$k = 20\frac{N}{m}$ $m = 0.5kg$

$\omega_0^2 = \frac{k}{m} = \frac{20}{0.5} = 40$ $\omega_0 = \sqrt{40} = 6.32 s^{-1}$

We expand a spring to a maximum ampletude of ~~0.5~~ 1m, and let go




after 10 minutes the max. amplitude has shrunk to 0.6m, Find b

$0.6 = 1e^{-\frac{b}{2m}600s}$

$0.6 = e^{-\frac{b}{1}\cdot 600s}$

$\omega_1 = \sqrt{\left(\frac{k}{m}\right)\omega_0^2 - \left(\frac{b}{2m}\right)^2}$ $\ln 0.6 = -600 \cdot b$