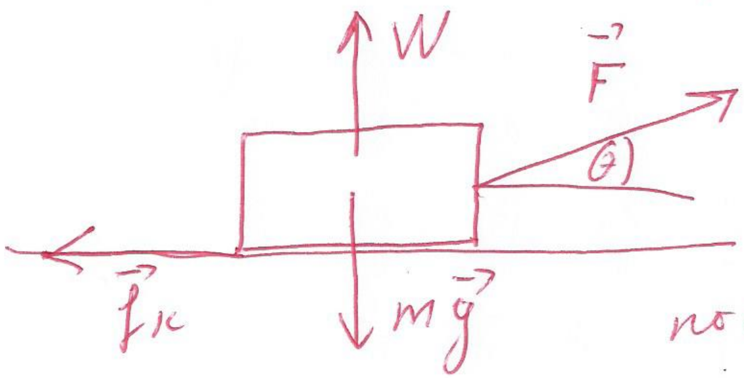
Work + energy
Ch. 7+8

Lecture 10

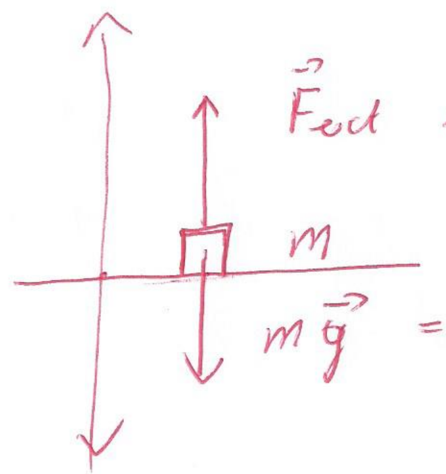
- Work with a constant force $\vec{F}$

$$W = \vec{F} \cdot \vec{r} = F_x \, x + F_y \cdot y$$
$$= F \cdot r \cdot \cos\theta$$



no acceleration

Work done by all exterior forces = 0



$$W_g = -mg(y_f - y_i) = -\Delta U$$

$W_{ext} = mgy$ } potential energy of gravitation ΔU

After lifting m by mgy

The potential energy $U(y)$ is defined in reference to a 0 point for the potential energy.

Such forces are called conservative forces.

The relationship between the work done by a conservative force and its potential energy U is

$$\boxed{W_{con} = -\Delta U}$$ always true

$$\vec{F}_g = -mg\vec{j} \qquad W = -mg\vec{j} \circ y\vec{j}$$

$$= -mgy = -\Delta U$$

$$\Delta U = mgy$$

Non constant force $\vec{F}_s = -kx\vec{i}$

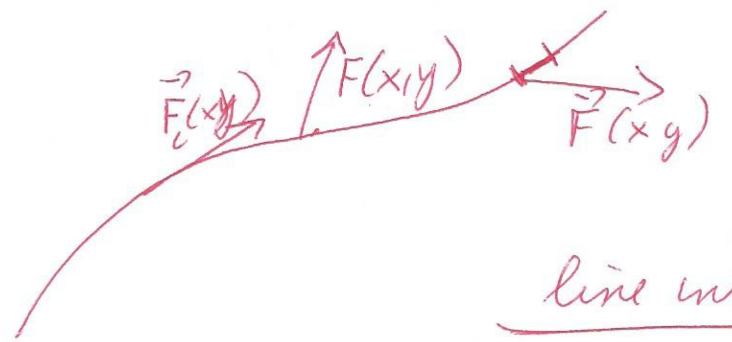
$$W = \int_{x_i}^{x_f} \vec{F}_s \circ d\vec{x} = \int_{x_i}^{x_f} -kx\,dx$$

$$= -\frac{1}{2}k(x_f^2 - x_i^2) = -\Delta U$$

$$\Delta U = \frac{1}{2}k(x_f^2 - x_i^2)$$

$U = \frac{1}{2}kx^2$; k_s is the springconstant.

Variable force in several dimensions and arbitrary path.

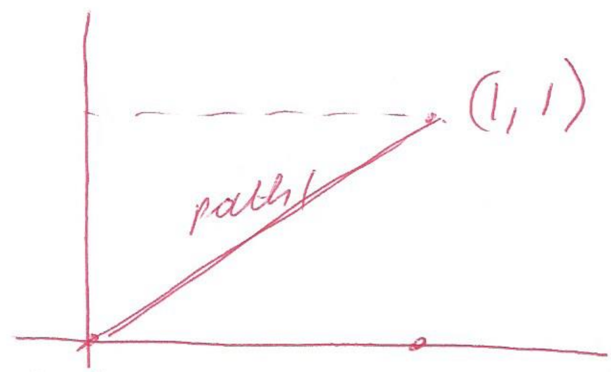


<u>line integral</u>

$$W = \int_{\text{path}} \vec{F} \cdot d\vec{r} = \int \left(F_x(x,y)\, dx + F_y(x,y)\, dy \right)$$

path   specify path.

<u>Example</u> : $\vec{F} = \langle xy, x^2 \rangle$



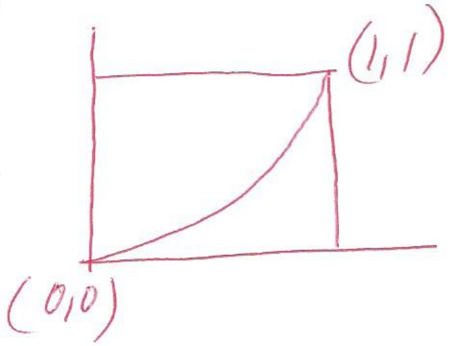
path 1 $y = x$

$$W = \int_{(0,0)\ y=x}^{(1,1)} xy\, dx + x^2 \cdot dy$$

$$= \int_0^1 (x^2 dx + x^2 dx) = \int_0^1 2x^2 dx = \frac{2x^3}{3} = \frac{2}{3}$$

p 4

$\vec{F} = \langle xy, x^2 \rangle$



$y = x^2$

$dy = 2x\,dx$

$$\int_{(0,0)}^{(1,1)} xy\,dx + x^2\,dy = \int_0^1 x \cdot x^2\,dx + \int_0^1 y\,dy$$

$$= \int_0^1 x^3\,dx + \int_0^1 y\,dy$$

$$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

Forces whose work done varies from path to path are non conservative, which means that no potential energy exists for them.

What we want for a conservative force mathematically speaking is a definition of work which leads to a simple integral:

$$\int \vec{F} \cdot \vec{dr} = \int (F_x\,dx + F_y \cdot dy) = \int_i^f dU =$$

$$= U_f - U_i$$

reminder : $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} \cdot dy$

$\int df = f$

$W = -\Delta U$

$\int (F_x dx + \int F_y dy)$

$\int \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} \cdot dy$

$F_x = \frac{\partial U}{\partial x} \quad F_y = \frac{\partial U}{\partial y}$

If $F_x = -\frac{\partial U}{\partial x} \quad F_y = -\frac{\partial U}{\partial y}$ then we are dealing with a conservative force.

$\vec{F} = -\vec{\nabla} U$

$\vec{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$ del operator.

$\vec{\nabla} \cdot U = \frac{\partial U}{\partial x} \vec{i} + \frac{\partial U}{\partial y} \vec{j}$

gradient.

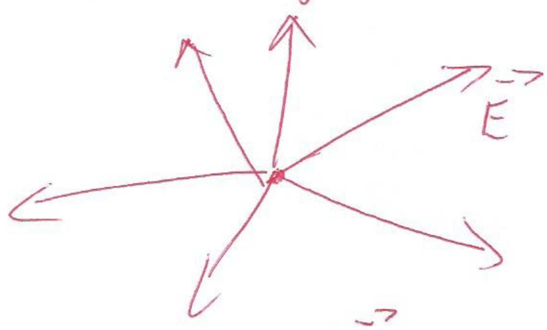
$$\vec{\nabla} \equiv \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$\vec{F} = -\vec{\nabla} \cdot U = \text{gradient of } U$$

$$\boxed{\vec{F} = -\vec{\text{grad}}\, U}$$ conservative force.

$\vec{\nabla} \cdot \vec{E}$ = divergence of the vector $\vec{E}$

$$= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \; ; \quad \text{div}\, \vec{E} = -\frac{\rho}{\varepsilon_0}$$



$$\mu_0 \vec{j} = \vec{\nabla} \times \vec{B} = \text{curl of } \vec{B}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} = \vec{C}$$

$$C_x = \frac{\partial}{\partial y} B_z - \frac{\partial}{\partial z} B_y$$

Find the curl of a conservative force $\vec{F}$

$$\vec{F} = -\vec{\nabla} U$$

$$\vec{\nabla} \times \vec{F} = \vec{\nabla} \times (-\vec{\nabla} U) = \vec{0}$$

$$\vec{F} = \langle F_x , F_y \rangle = \langle F_x(x,y) , F_y(x,y) \rangle$$

You only need to calculate the 3rd component

$$C_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \overset{?}{=} 0$$

$$\vec{F} = \langle x y , x^2 \rangle$$

$$\frac{\partial x^2}{\partial x} - \frac{\partial (xy)}{\partial y} = 2x - x$$

$$\vec{F} = \langle 2xy , x^2 \rangle$$

$$(\text{curl } \vec{F})_z = 2x - 2x = 0$$

$$\int 2xy \, dx + x^2 \, dy$$

$$\vec{F} = - \overrightarrow{\text{grad}} \, V$$

$$\boxed{U = x^2 y}$$

Conservative force $\vec{F}$:

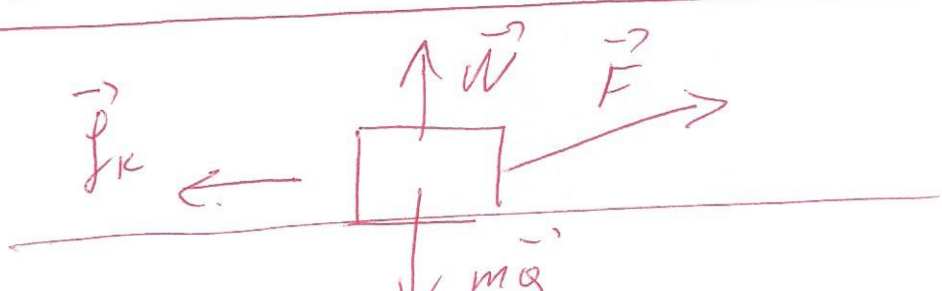
a) path indendent

b) $\vec{F} = -\,\overrightarrow{\text{grad}}\, U$

c) $W = -\Delta U$

d) $\vec{\nabla} \times \vec{F} = \vec{0}$

Work when velocity changes



Assume that we have ~~const~~ acceleration

$$W = \vec{F} \cdot \vec{r} = F_x \cdot x$$

$$= m\, \underline{a \cdot x}$$

Constant acceleration

$$v_f^2 = v_0^2 + 2a(x_f - x_0)$$

$$a(x_f - x_0) = \frac{v_f^2 - v_0^2}{2}$$

$$W = \frac{m}{2}\left(v_f^2 - v_0^2\right) = \Delta K \; ; \; K = \frac{1}{2} m v^2$$

General situation:

$$W = \int_{\substack{\text{path} \\ \text{point 1 to point 2}}} \sum \vec{F} \cdot d\vec{r} \qquad \sum_{ext} \vec{F} = m\vec{a}$$

$$= \int m\vec{a} \cdot d\vec{r} \quad ; \quad \vec{a} = \frac{d\vec{v}}{dt}$$

$$d\vec{r} = \vec{v}\, dt$$

$$= \int_{\vec{v}_i}^{\vec{v}_f} m \frac{d\vec{v}}{\cancel{dt}} \cdot \vec{v}\, \cancel{dt} =$$

$$\int \vec{v}\, d\vec{v} = \int v_x \cdot dv_x + \int v_y \cdot dv_y$$

$$= \frac{v_x^2}{2} + \frac{v_y^2}{2} = \frac{\vec{v}^2}{2}$$

$$W = \frac{1}{2} m \vec{v}^2 \Big|_{v_i}^{v_f} = \Delta K$$

$$\boxed{W = \Delta K}$$