34 Electromagnetic Waves from Maxwell's Equations .

For homework, see website: 1,5,7,11,13,17,19,27,29

Summary and abstract:

The instantaneous energy density u(t) in an electromagnetic field is given by the sum of the energy densities we derived for a capacitor and for a coil. The electric and magnetic fields are in general functions of all space coordinates and time.

$$u(t) = \frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{1}{2\mu_0}\vec{B}\cdot\vec{B}$$

(34.1)

$$\vec{E} = \vec{E}(\vec{r},t) = \vec{E}(x,y,z,t); \vec{B} = \vec{B}(\vec{r},t) = \vec{B}(x,y,z,t)$$

Table of Content:

34.1 Derivation of the electromagnetic wave equations

For an electromagnetic wave there are no currents and no charges and we get the following Maxwell equations for the vacuum:

(34.2)

$$a)\nabla\times\vec{B} = \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \text{ and b) } div\vec{B} = 0$$

$$c)\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \text{ and d) } div\vec{E}=0$$

We see, that any time changing electric field is surrounded by a time changing magnetic field, any time changing magnetic field is surrounded by a time changing electric field. This implies that any accelerated charge is surrounded by both electric and magnetic fields which both change in time.

These combined fields give rise to an electromagnetic wave, which consists of time changing magnetic and electric fields, perpendicular to each other and traveling in a direction perpendicular to both at the speed of light c. We shall see how this follows from Maxwell's equations.

We are going to show that both the **electric field and the magnetic field satisfy a wave equation just like the wave equation for a wave on a string**. There, we derived the **linear wave** equation for a one dimensional wave on a string: y(x,t) is the linear wave-function which obeys the (partial differential) wave equation:

(34.3) $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2 y}{\partial t^2};\; y = y(x,t)$$

One way to write its solution is in terms of an exponential complex function. You will recognize the same exponent $i\omega t$ which we also had for ac-currents and voltages.

(34.4) $$y(x,t) = y_{\max} e^{i(kx-\omega t)};\; \mathrm{v} = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f = c$$

In contrast to a linear wave, the electric and the magnetic fields and waves have three components, each of which is a function of x,y,z,t (in Cartesian coordinates).

(34.5) $$\vec{E}(x,y,z,t) \; and \; \vec{B}(x,y,z,t)$$
$$\vec{E} = E_x\vec{i} + E_y\vec{j} + E_x\vec{k} \; and \; \vec{B} = B_x\vec{i} + B_y\vec{j} + B_x\vec{k}$$

Every component satisfies a wave equation like (34.3), which is what we are going to show.
We start with the assumption that **we are in a region of space where there are no charges and currents, but where time varying electric and magnetic waves are present**. Such a region of space is referred to as the vacuum. None of the Maxwell equation contain charge or current densities and we are left with the four equations:

(34.6) $$a)\nabla\times\vec{B} = \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \text{ and b) } div\vec{B} = 0$$
$$c)\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \text{ and d) div}\vec{\mathrm{E}}=0$$

we see that if we apply the differential operator , $curl = \vec{\nabla}\times$, to the first equation (34.6) a) we get an expression with $\vec{\nabla}\times\vec{E}$ on the right side, like this

(34.7) $$\vec{\nabla}\times\left(\nabla\times\vec{B} = \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}\right)$$

(34.8) $$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right) = \varepsilon_0\mu_0\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{E}\right) = \varepsilon_0\mu_0\left(\vec{\nabla}\times\frac{\partial\vec{E}}{\partial t}\right)$$

We use Faraday's law on the right side:

(34.9) $$\varepsilon_0\mu_0\left(\underbrace{\vec{\nabla}\times\frac{\partial\vec{E}}{\partial t}}_{=-\frac{\partial\vec{B}}{\partial t}}\right)=\varepsilon_0\mu_0\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right)=-\varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t^2}$$

On the right side we now have a constant times the second time derivative of the magnetic field.

What the hell is $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right)$?

We have still to evaluate the left side of (34.8) with the double curl expression. Now, remember that the curl is both a vector and a derivative operator, which means that we have to apply the product rule for derivation, maintaining the correct order of the cross product which is anti commutative. Also, remember how we can convert a double cross product according to

(34.10) $\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}\left(\vec{A}\bullet\vec{C}\right)-\vec{C}\left(\vec{A}\bullet\vec{B}\right)$ for any three vectors $\vec{A},\vec{B},\vec{C}$

(See: 230 ch19supp2 Vector operators.docx)

We apply the double curl rule on (34.2) c)

(34.11) $$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right)=\vec{\nabla}\left(\vec{\nabla}\bullet\vec{B}\right)-\vec{B}\left(\vec{\nabla}\bullet\vec{\nabla}\right)=\vec{\nabla}\left(\underbrace{div\vec{B}}_{=0}\right)-\vec{\nabla}^2\vec{B}=-\vec{\nabla}^2\vec{B}\Rightarrow$$

On the right side we have now the scalar product of the del operator with itself:
(34.12)

$$\vec{\nabla}\bullet\vec{\nabla}=\vec{\nabla}^2=\left\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\rangle\bullet\left\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\rangle=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\Delta \text{ called the Laplace operator}$$

which is a scalar derivative operator.

(We encountered this operator before in the derivation of the Biot-Savart law ch29 magnetic fields.docx).
The first term of this expression is the second partial derivative with respect to x, which was also part of the linear wave-equation. This means that we have just obtained the wave equation for the magnetic field:

(34.13) $$\boxed{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\vec{B}=\Delta\vec{B}=\varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t^2}}$$

We can immediately see that the factor on the right hand side must be equal to the speed of propagation of the electromagnetic wave:

(34.14) $$\boxed{\varepsilon_0\mu_0=\frac{1}{c^2}}$$

As this is a vector equation it represents a scalar equation for each component of the magnetic field.

(34.15) $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)B_x\left(x,y,z,t\right)=\varepsilon_0\mu_0\frac{\partial^2 B_x\left(x,y,z,t\right)}{\partial t^2}$$

and so on for each component of the magnetic field

By applying the curl operator on equation c) in (34.6) we get the wave equation of the electric field:

(34.16) $$\boxed{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\vec{E}=\Delta\vec{E}=\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}}$$

Each component of these vector fields satisfies the wave equation.
For example, the wave equation for B_x is then, :
(34.17)

$$a)\vec{\nabla}^2 B_x\equiv\frac{\partial^2 B_x}{\partial x^2}+\frac{\partial^2 B_x}{\partial y^2}+\frac{\partial^2 B_x}{\partial z^2}=\varepsilon_0\mu_0\frac{\partial^2 B_x}{\partial t^2}$$

the same equation holds also for all three components of the electric field

$$b)\vec{\nabla}^2 E_x\equiv\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}=\varepsilon_0\mu_0\frac{\partial^2 E_x}{\partial t^2}$$

34.2 Solution of the wave equation for e.m. waves:

The same mathematical equations have the same solutions, therefore we can write down the solution for each of the components of the electric and magnetic fields following what we learned when discussing the linear wave equation for waves on a string or sound waves:
For a linear one dimensional wave we had in complex notation:

(34.18) $$y(x,t)=y_{\max}e^{i(kx-\omega t)}$$

Here the **wave number** k $=2\pi/\lambda$ was associated with a wave propagating in the x-direction. It is easy to generalize the wave number in one direction to a wave number vector

$$\vec{k} = \langle k_x, k_y, k_z \rangle ; k_x = \frac{2\pi}{\lambda_x}$$

(34.19) where the wavelength is the wavelength of the wave propagating in the x direction.

$$\vec{k} \bullet \vec{r} = k_x x + k_y y + k_z z$$

Thus, we can easily generalize the linear wave into a spatial wave spreading in all directions:

(34.20) $$y(x,y,z,t) = y(\vec{r},t) = y_{\max} e^{i(\vec{k}\vec{r} - \omega t)}$$

When we do the same for each component of the magnetic and the electric fields we can immediately write down the general form of their solutions:

(34.21)
$$\vec{E}(x,y,z,t) = \vec{E} = \vec{E}_{\max} e^{i(\vec{k} \bullet \vec{r} - \omega t)}$$
$$\vec{B}(x,y,z,t) = \vec{B} = \vec{B}_{\max} e^{i(\vec{k} \bullet \vec{r} - \omega t)}$$

34.3 Derivatives $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ of the wave functions in exponential form:

As we saw already in the case of the time-derivatives of the complex functions for charges and currents in the previous chapter, the derivatives of exponential functions are particularly easy to perform,

(34.22)
$$E_x(x,y,z,t) = E_0 e^{i(k_x x + k_y y + k_z z - \omega t)}$$
$$\frac{\partial}{\partial t} E_x = E_0 \frac{\partial}{\partial t} e^{i(k_x x + k_y y + k_z z - \omega t)} = -i\omega E_x$$
$$\frac{\partial}{\partial x} E_x = E_0 \frac{\partial}{\partial x} e^{i(k_x x + k_y y + k_z z - \omega t)} = E_0 i k_x e^{i(k_x x + k_y y + k_z z - \omega t)} = i k_x E_x$$

34.3 a Operator derivatives $\vec{\nabla} \bullet$ and $\vec{\nabla} \times$ applied to exponential functions:

The del operator $\vec{\nabla}$ applied to an exponential function of the form $e^{i(\vec{k} \bullet \vec{r} - \omega t)}$ turns into a multiplication with the vector $i\vec{k}$. Just like the partial time derivative turns into a multiplication with $i\omega$, so does the del operator turn into a multiplication with $i\vec{k}$

(34.23)
$$\frac{\partial}{\partial t}(-i\omega t) = -i\omega$$
$$\vec{\nabla} \Rightarrow i\vec{k}$$

In the case of the del operator, we must distinguish the ways in which the derivative operates, namely, as a scalar product, or as a cross-product:

(34.24)
$$\vec{\nabla} \times \vec{B} = curl\vec{B} = i\vec{k} \times \vec{B}; \vec{\nabla}\vec{B} = div\vec{B} = i\vec{k} \cdot \vec{B}$$
$$\vec{\nabla} U = \overrightarrow{grad} U = i\vec{k} U$$
$$\vec{\nabla} \bullet \vec{\nabla} = \Delta = Laplace = (i\vec{k})^2 = -k^2$$

34.3b Using the new rules on e.m. wave functions:

(34.25)

$$\vec{E}(x,y,z,t)=\vec{E}=\vec{E}_0 e^{i\left(\vec{k}\bullet\vec{r}-\omega t\right)}$$
$$\vec{B}(x,y,z,t)=\vec{B}=\vec{B}_0 e^{i\left(\vec{k}\bullet\vec{r}-\omega t\right)}$$

Maxwell's equations (34.6) can be easily expressed by using the derivative rules above. We obtain a series of equations which help us understand the peculiar behavior of electromagnetic waves.

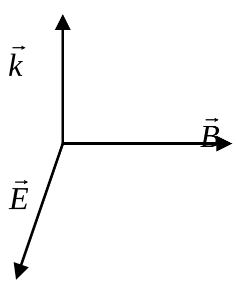
Maxwell's equations in the vacuum turn into products between vectors. Let us start with

(34.26)
$$div\vec{E}=0\Rightarrow i\vec{k}\cdot\vec{E}=0\Rightarrow\vec{E}\perp\vec{k}$$
$$div\vec{B}=0\Rightarrow i\vec{k}\cdot\vec{B}=0\Rightarrow\vec{B}\perp\vec{k}$$

This means that the divergence rules for the electric and magnetic field have as a consequence that the directions of the electromagnetic fields in an e.m. wave are always perpendicular to the direction of propagation, determined by the direction of wave-number vector $\vec{k}$.

(34.27)
$$curl\vec{E}=-\frac{\partial\vec{B}}{\partial t}\Rightarrow i\vec{k}\times\vec{E}=-\left(-i\omega\right)\vec{B}\Leftrightarrow\vec{k}\times\vec{E}=\omega\vec{B}$$

This means that $\vec{E}$ is also perpendicular to $\vec{B}$. We apply the right hand rule to determine the relative directions of the three vectors, $\vec{E},\vec{B},\ and\ \vec{k}$.



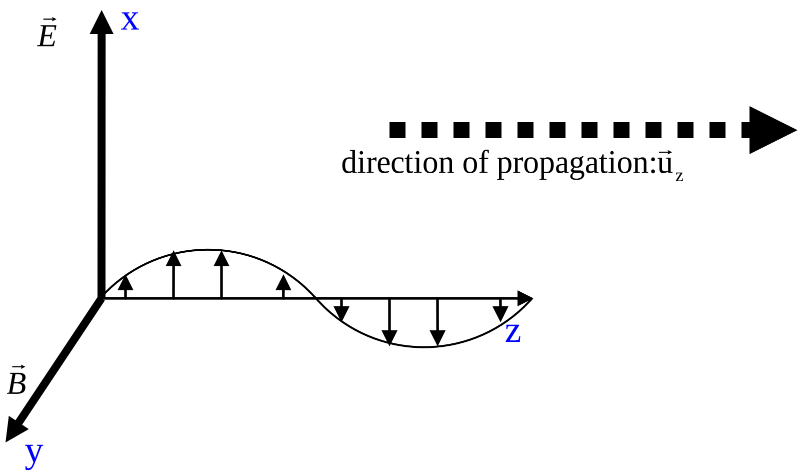
By just considering the magnitudes in (34.27) we see that:

(34.28) $E=\frac{\omega}{k}B=\frac{\lambda}{T}B=\lambda\nu B=cB$

(34.29) $\boxed{E=cB}$

We also see that the electric and magnetic fields in an e.m. wave are always in phase.(There is no imaginary number in the final equation (34.27), which would cause a phase shift between the electric and the magnetic field.)

The right hand rule determines the relative direction of the electric with the magnetic field, both of which are perpendicular to the direction of propagation.



The relative orientation of the three directions of $\vec{E}, \vec{B}$, and $\vec{k}$ which are perpendicular to each other is such that the cross-product $\vec{E} \times \vec{B}$ points in the direction of propagation $\vec{k}$.

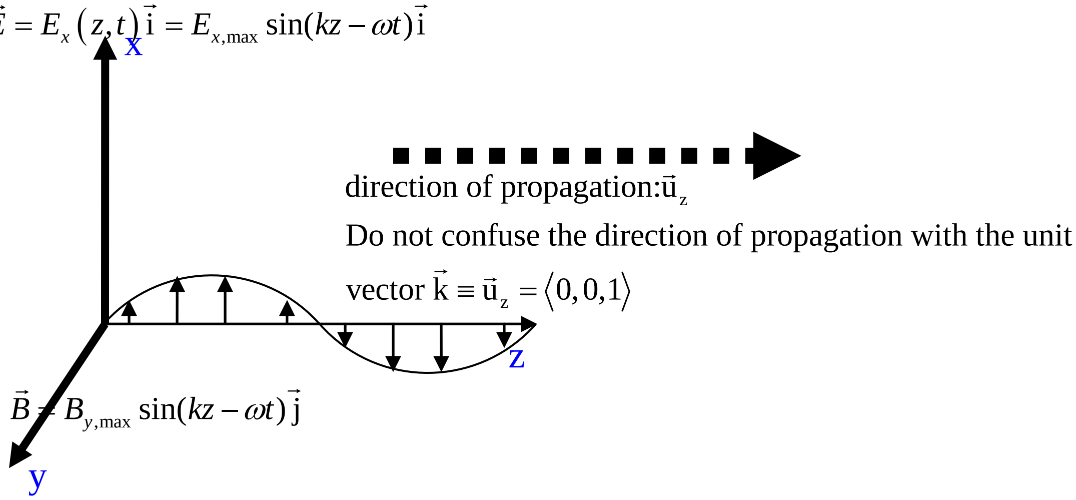
Here is an example: Draw the picture of a plane e.m. wave propagating in the z-direction. Suppose that the wavelength is 50.0cm, and the electric field vibrates in the x-z plane with an amplitude of 220V/m. a) calculate the frequency of the wave and b) the magnitude and direction of $\vec{B}$ when the electric field has its maximum value in the positive x direction c) write an expression for $\vec{B}$ with the correct unit vector with numerical values for $B_{max,}$ k, and angular frequency, and with its magnitude in the form

$$B = B_{\max} \sin\left(kz - \omega t\right)$$

a) fλ=3E8 gives us f=6E8Hz; b) $B_{max} = \frac{E_{\max}}{c} = 7.33 \cdot 10^{-7} T$ in the –y direction c)

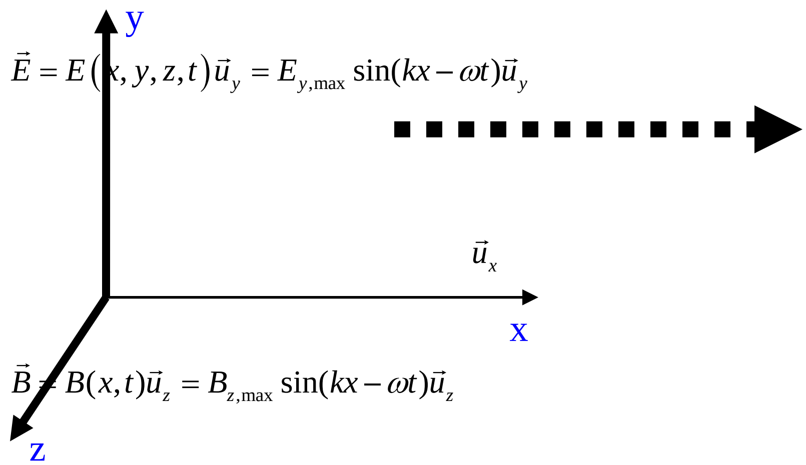
$$\vec{B} = -7.33 \cdot 10^{-7} T \sin(\frac{2\pi}{0.5m} z - 12\pi \cdot 10^{8} t)\vec{\mathrm{j}} \text{ or } \vec{B} = -7.33 \cdot 10^{-7} T e^{i(6\pi z - 12\pi E8t)}\vec{\mathrm{j}}$$

$$\vec{E} = E_x\left(z,t\right)\vec{\mathrm{i}} = E_{x,\max} \sin(kz - \omega t)\vec{\mathrm{i}}$$



To better study the behavior of the e.m. wave we assume that the propagation occurs in the x-direction, and that $\vec{E} = E_y e^{i(kx-\omega t)}\vec{\text{j}} = E(x,t)\vec{\text{j}}$ Consequently, we have fixed the magnetic field in the z direction $\vec{B} = Be^{i(kx-\omega t)}\vec{u}_z$ **Such electromagnetic waves in which $\vec{B}$ and $\vec{E}$ are confined to a fixed direction perpendicular to the direction of propagation are called linearly polarized**.

For actual calculations with real numbers we use $\sin(kx-\omega t)$ or $\cos(kx-\omega t)$.



Now, in contrast, assume a **circularly polarized** e.m. wave, which, at some point in time, has the following electric field with components in the x- and y- directions, associated with it: $\vec{E}(z,t) = \vec{E}_1 + \vec{E}_2 = E_0\sin(kz-\omega t)\vec{\text{i}} + E_0\cos(kz-\omega t)\vec{\text{j}}$
Find the corresponding magnetic field as well as the scalar and cross-products of the electric and magnetic fields. This field oscillates in the x-z and the y-z planes. So does the magnetic field. The wave propagates in the z-direction. The first component of B must point in the $\vec{\text{j}}$ direction, the second component in the $-\vec{\text{i}}$ direction, so that the cross-products both point in the +z direction. $\vec{\text{i}} \times \vec{\text{j}} = \vec{\text{j}} \times \left(-\vec{\text{i}}\right) = \vec{\text{k}} \equiv \vec{\text{u}}_z$; $\vec{\text{k}} = \langle 0,0,1 \rangle$

$\vec{B}(z,t) = \vec{B}_1 + \vec{B}_2 = B_0\sin(kz-\omega t)\vec{\text{j}} - B_0\cos(kz-\omega t)\vec{\text{i}}$

Therefore $\vec{E}\cdot\vec{B} = 0$ and $\vec{E}\times\vec{B} = \left[E_0B_0\sin^2(kz-\omega t) + E_0B_0\cos^2(kz-\omega t)\right]\vec{u}_z = E_0B_0\vec{u}_z$

The electric and magnetic field vectors are always perpendicular to each other, but they rotate in their plane with an angular velocity ω.

34.4 The Poynting vector, electromagnetic energy intensity vector:

The direction of propagation of any e.m. wave it is also the direction in which energy is being carried. The intensity of the electromagnetic wave is given by the Poynting vector .

(34.30) $$\vec{S} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)$$

Next we are going to study the relationship between this energy intensity vector, the Poynting vector, which is a vector function $\vec{S}(r,t)$ and the energy density $u(r.t)$, which is a scalar function.

$\vec{S} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)$ Poynting vector, instantaneous intensity vector

(34.31) of an electromagnetic field = power per unit area. $\left[\mathrm{S}\right] = \frac{Watts}{m^2}$

Let's just look at the magnitudes:

(34.32) $$u = \frac{\varepsilon_0\vec{E}^2}{2} + \frac{\vec{B}^2}{2\mu_0} = \frac{\varepsilon_0 c^2 B^2}{2} + \frac{\vec{B}^2}{2\mu_0} = \frac{B^2}{\mu_0}; with\ \varepsilon_0\mu_0 c^2 = 1$$

(34.33) $$\left|\vec{S}\right| = \frac{EB}{\mu_0} = \frac{cB^2}{\mu_0} = uc$$

We recognize in $\frac{B^2}{\mu_0}$ the magnetic **energy density** u. So, S has the units of energy density times m/s which is: $\frac{J}{m^3}\frac{m}{s} = \frac{J}{m^2 s} = \frac{Power}{area} = \frac{energy}{time\cdot area}$ =intensity

Let us look at it differently: u is the energy density. If we multiply u by the volume ΔV=AΔx we get the total energy contained in the volume ΔV. This volume is carved out by the Poynting vector in the time Δt. The em wave travels at the speed of light, so:

(34.34) $$dU = u\cdot dV = u\cdot A\cdot dx = u\cdot A\cdot cdt \Rightarrow$$
$$\frac{dU}{dt} = power = u\cdot A\cdot c = \frac{S}{c}A\cdot c = S\cdot A$$

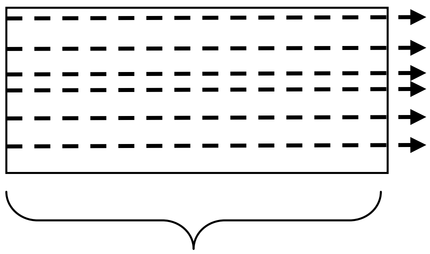
The Poynting vector $\vec{S}$ defines the **energy current density** of an electromagnetic field. While this current density vector travels through any volume in space it carries energy (power) with it. When it passes through an arbitrary cross-section A for a time dt, an energy stream passes through that cross-section which has the value S·A·dt. The electromagnetic energy which was in the volume Adx=Avdt is uAvdt, with v=c, the speed of light. Thus, the energy content of the volume **decreases**. The energy stream through the cross-section A in the time dt, decreases the energy of the volume. This is most elegantly expressed as:

(34.35) $$div\vec{S} = -\frac{\partial u}{\partial t}$$

If we apply Gauss' law:

(34.36)
$$\iiint_V div\vec{S}dV = -\iiint_V \frac{\partial u}{\partial t}dV = -\frac{d}{dt}\iiint_V udV = -\frac{dU}{dt}$$
$$\oiint_{\partial V} \vec{S}d\vec{A} = -\frac{dU}{dt}$$

The flux of the instantaneous electromagnetic intensity through a closed surface is equal to the power leaving the volume contained in the surface.



$\vec{S}\cdot\vec{A}$ =is the rate of decrease of total energy = -$\frac{dU}{dt}$

$$dU = udV = Adx = Acdt$$

(34.37)
$$S\cdot Adt = uAdx \Rightarrow SA = uA\frac{dx}{dt} = uAc$$

(34.38)
$$\boxed{S = uc}$$

The time **averaged** Poynting vector and the time **averaged** energy density over a whole period is given by assigning an rms value to the amplitude of the magnetic field and the electric field. As both, the electric field and the magnetic field in an e.m. wave vary like a sine function, the average value over a whole period is equal to ½, just like in the case of the average power of an ac-current.

(34.39)
$$E_{rms} = \frac{E_{\max}}{\sqrt{2}}; B_{rms} = \frac{B_{\max}}{\sqrt{2}}$$

(34.40)
$$\bar{S} = \frac{1}{\mu_0}\frac{E_{\max}}{\sqrt{2}}\frac{B_{\max}}{\sqrt{2}} = \frac{1}{2}S_{\max} \text{ and } \bar{u} = \frac{1}{2}u(t)$$

The average value of the Poynting vector is the average intensity=average power/unit area.

(34.41)
$$\bar{S} = \frac{1}{2}\frac{E_{\max}^2}{\mu_0 c} = \frac{1}{2}\frac{cB_{\max}^2}{\mu_0} \equiv I = \text{intensity}$$

For the average energy density of an electromagnetic wave this means obviously that:

(34.42)
$$\boxed{\bar{u} = \frac{\varepsilon_0}{2}E_{\max}^2 = \frac{B_{\max}^2}{2\mu_0} = \frac{\bar{S}}{c}}$$

We arrive at the same result formally, if we use the fact that the Poynting **vector** $\vec{S}$ (energy intensity=power/area) behaves mathematically like an electromagnetic wave. We assume the same is true for the energy density u. Thus, we can write:

(34.43) $$\vec{S} = \vec{S}_{\max} e^{i(\vec{k}\bullet\vec{r}-\omega t)}; u = u_{\max} e^{i(\vec{k}\bullet\vec{r}-\omega t)}$$

We derived the energy conservation of an e.m. wave in a more intuitive fashion earlier, namely, that the flux of the e.m.-intensity vector $\vec{S}$ through a surface takes the energy of the wave with it. In magnitudes, this just becomes $SA = \frac{dU}{dt}$. The accurate mathematical relationship is called a continuity equation for e.m. waves:

(34.44) $$div\vec{S} = -\frac{\partial u}{\partial t}$$

From (34.26) we know how to perform the derivatives on an exponential wave:

(34.45) $$div\vec{S} = i\vec{k}\cdot\vec{S} \text{ and } \frac{\partial u}{\partial t} = -i\omega u$$

Therefore, the continuity equation for the Poynting vector (34.44), can be said to express the fact of **local** energy conservation of electromagnetic waves. As the wave travels through space it does not leave energy behind; the energy it carries through any cross-section during a certain time is exactly the energy contained in the volume through which it traveled during this time.

(34.46) $$div\vec{S} = -\frac{\partial u}{\partial t} \Rightarrow i\vec{k}\cdot\vec{S} = -(-i\omega)u$$

The direction of propagation is parallel to the direction of the Poynting vector, therefore:

(34.47) $$\boxed{kS = \omega u \Rightarrow S = \frac{\omega}{k}u = cu}$$

Problem: calculate the average energy contained in a unit volume through which an e.m. wave passes, whose maximum electric field is 250V/m.

$$\bar{U} = \frac{1}{2}uV = \frac{1}{2}\frac{S_{\max}}{c}V = \frac{E_{\max}^2}{2\mu_0 c^2}V = \frac{\varepsilon_0}{2}E_{\max}^2 V = \frac{8.85\cdot 10^{-12}}{2}250^2\cdot 1 = 0.277\mu J$$

Note:

(34.48)
$$\textit{Intensity} \text{ of an em wave}: S_{avg} = u_{avg}c$$
$$\textit{Power} \text{ of an em wave}: S_{avg}A$$
$$u_{avg} \equiv \bar{u} = \frac{u_{\max}}{2} = \frac{\varepsilon_0 E^2_{\max}}{2}$$

34.5 Pressure in an electromagnetic wave:

If an e.m. wave hits a surface the energy contained in the wave gets absorbed by the surface and is partially reflected back. If there is no reflection (totally black surface) we talk about a perfect absorber. If all energy is reflected (mirror) we talk about a perfect reflector. (In reality there is no such thing as an absolutely perfect reflector or absorber, they are just limiting cases to consider).

By writing the energy density in terms of the total energy divided by the volume, we see that energy density can also be interpreted as the *pressure* of the Poynting vector on a surface A during the time *t in which it carves out the volume V*. These derivations are not rigorous, but the results are correct.

(34.49)
$$S = uc$$

(34.50)
$$\boxed{\frac{S}{c} = u = \frac{U}{V} = \frac{Fdx}{Adx} = pressure\ P}$$

Problem: A 100mW laser beam is reflected from a mirror. Calculate the force on the mirror.
Force is change of momentum over time. Pressure is force over area=F/A=u=S/c.

$$F = \frac{S}{c}A = \frac{power}{Ac}A = \frac{power}{c}$$

For complete reflection we need twice the force of perfect absorption. F=2P/c=200mW/c.

<u>34.6 Momentum of an electromagnetic wave</u>: This again leads us to the *amount of momentum exchanged* with the surface *A*, by writing *F=dp/dt.*
While the e.m. wave travels through the volume *Adt* it can be considered to impart an average momentum onto the surface, which is the momentum contained in the e.m. wave of volume V. If we actually place a black surface into the path of the wave, all energy enters the black surface, none is reflected. Thus the total momentum of the wave contained in the volume V gets imparted to that surface.

(34.51)
$$\Pr essure = \frac{F}{A} = \frac{dp}{dt}\frac{1}{A} = u; dp = u \cdot dt \cdot A$$
$$dp = u \cdot dt \cdot \frac{Adx}{dx} = \frac{uV}{c} = \frac{U}{c}$$

(34.52)
$$\boxed{dp \Rightarrow p = \frac{U}{c} \text{ perfect absorber}}$$

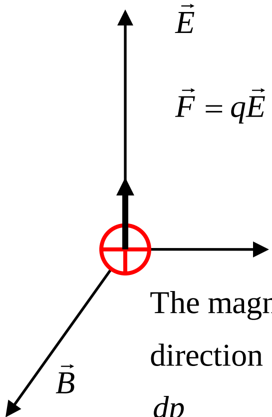
Momentum crossing the section A in the time *dt* is equal to the *total energy contained in the volume carved out by the wave front in the time dt*, divided by c. This is not only true for infinitesimal volumes but for any volumes due to $\frac{dx}{dt} = \frac{x}{t} = c$ Therefore we are justified in using finite momentum p:
The formula (34.52) is for perpendicular incidence of the light energy on a perfectly absorbing surface. Now, for a perfect reflector the change of momentum before and after being reflected from the surface is twice the result in (34.52)

(34.53)
$$\boxed{|dp| = |p_{out} - (-)\,p_{in}| = 2p = \text{total momentum} = \frac{2U}{c} \text{ Perfect reflector}}$$

<u>Note that the momentum imparted to the surface is equal to the momentum of the e.m. wave</u>. A reflected wave carves out twice the volume which leads to twice the energy and twice the momentum.

(34.54) $p = \frac{U}{c}$ Perfect absorber

Let us see if we can understand this on a more basic level. We ask what happens to a charge q on the surface when an e.m. wave hits it. Assume a linearly polarized wave with E in the x direction and B in the y direction. The direction of propagation is z.



$$\vec{F} = q\vec{E} = m\vec{a} \Rightarrow \text{v=at=}\frac{\text{qE}}{\text{m}}t \Rightarrow K = \frac{1}{2}m\text{v}^2 = \frac{1}{2}(qEt)^2$$

The magnetic field exerts a force on the moving charge, in the direction of propagation: $\vec{F}_B = q\vec{v} \times \vec{B} \Rightarrow$

$$\frac{dp}{dt} = q\text{vB} = \frac{\text{q}^2 EB}{m}t \Rightarrow p = \frac{1}{2}\frac{\text{q}^2 EB}{m}t^2 = \frac{1}{c}(qEt)^2; with: B = \frac{E}{c}$$

With this crude model we can also see that the momentum of an electromagnetic wave is equal to the total energy of the wave divided by c.

(34.55) $$p = \frac{U}{c}$$

We use U for potential energy (instead of E) so as not to create confusion with the electric field E.

Food for thought (if time permits): Optional

34.7 Quantum-physical momentum and energy of a photon.

If we probe deeper into the phenomenon of electromagnetic waves we enter the field of quantum mechanics, and we encounter the difficult issue of wave particle duality. Electromagnetic waves actually consist of particles, photons, which have momentum and energy associated with them and spin. For every particle we have the always correct quantum-physical formulas:

(34.56)
$$\vec{p} = \hbar\vec{k} \text{ and energy E=}\hbar\omega$$
$$energy = E = \sqrt{p^2c^2 + m_0^2c^4}$$

As a photon does not have a rest mass, m_0=0 and we get the same formula for momentum we just arrived at, namely:

(34.57) $$E \equiv U = pc$$

which is the same result we get when calculating the speed of propagation of the e.m. wave:

(34.58) $$\boxed{c=\frac{\omega}{k}=\frac{\hbar\omega}{\hbar k}=\frac{E}{p}}$$

Energy $E=\hbar\omega$ and momentum $\vec{p}=\hbar\vec{k}$ of a magnetic wave are really the quantities behind the angular frequency and wave-number vector.
Energy and momentum of a photon are related to the wavelength and frequency of the corresponding electromagnetic wave through equation (34.58).
Earlier on, when we talked about how the sun creates its energy we saw that in the process of fusing four protons into a Helium nucleus (α-particle=nnpp) 17.6 MeV of energy is being released during each process. This energy is the energy of photons being created during this process. Let us calculate their wavelength and frequency and compare them to the wavelength and frequency of visible light with the wavelength of 600nm.

$$17.6MeV=17.6\cdot 1.6\cdot 10^{-13}J=\hbar\omega=hf$$

(34.59) $$\omega=\frac{17.6\cdot 1.6\cdot 10^{-13}J}{1.055\cdot 10^{-34}Js}=2.67\cdot 10^{22}\frac{1}{s}\text{ or } f=4.25\cdot 10^{21}Hz$$

Electromagnetic waves with such a high energy (frequency) are called γ-rays. They easily penetrate human tissue and just about everything else and destroy molecular structures. Thus, they are highly dangerous to all living things. The sun does not emit such radiation, thank you God, which means that the photons created in the interior of the sun where we have temperatures of 1 million K and more, must lose most of their energy while they are traveling to the surface of the sun. We can understand this again through the concepts of mean free path. The photons bump into protons, neutrons, and α-particles, and move towards the surface very slowly, which is 696 Mm away, 432,000 miles (109 times the radius of the earth. (The sun is roughly 1 million times larger than the earth!). The concentration of particles is highest in the center of the Sun, therefore there is a gradient of density from the interior to the exterior, resulting in a diffusion current towards the outside. On the other hand, gravity pulls the particles to the center. The resultant current is a current that allows the photons to slowly move to the outside of the sun.
If we are given the energy in an electromagnetic wave we can calculate the number of photons contained in it. Both energy and momentum of the total magnetic wave are multiples of the individual energies and momenta of individual photons. To describe the motion of individual particles or photons, we must use the complex exponential functions. For example, for a free particle we describe its motion through a function like:

(34.60) $$\psi=\psi_0 e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{r}-Et)}$$

which is a solution of the complex differential equation know as the Schrödinger equation:

(34.61) $$i\hbar\frac{\partial\psi}{\partial t}=H\psi$$

which can be interpreted as saying that the imaginary change in time of the function ψ is proportional to its energy, which carries the letter H. It is not simply an energy but a differential energy operator similar to the Laplace and del operators we encountered in this chapter.

This describes the reality of all particles in the universe. Imaginary numbers and imaginary waves of particles form the mysterious actual world of Maya: illusory reality-actuality or real-actual illusion, neither, and both. The only real, i.e. measurable part of this turns into statements of probabilities, which is very unsatisfactory and disconcerting for the human mind which wants certainty above all else.

Going back to the relationship between momentum and energy of an e.m. wave we memorize that the momentum transfer (for perpendicular incidence) of an electromagnetic wave onto a perfectly absorbing surface (black) is equal to the **total energy** in the wave (*in a particular volume of space traversed by the e.m. wave-front*) divided by the speed of light c. (34.55) p=U/c. For a perfect reflector like a mirror, this amount needs to be doubled.

If incidence in not perpendicular but makes an angle of θ with the perpendicular direction on the surface we have:

(34.62) $$p = \frac{2U}{c}\cos^2\theta \text{ } perfect\ reflector$$

Problem: A 3.0mW laser beam hits an area of 2mm radius. The Poynting vector (intensity) multiplied by the area gives us this average power:

$$SA = 3mW \Rightarrow S = \frac{3mW}{\pi r^2} = 955\frac{W}{m^2};$$

$$\frac{S}{c} = u = pressure$$

Problem: During a spacewalk an astronaut's tether to the space ship suddenly unhooks. Fortunately, he carries a 1kW emergency laser with himself. If he finds himself 25 m from the space ship, how long will it take him to get back, using the laser, if his total mass is 95.0kg, including the laser.

If he fires the laser in a direction opposite to the perpendicular surface of the space ship, we can use momentum conservation. The momentum of the laser light will equal his own momentum.

$$p = \frac{U}{c} \Rightarrow \frac{dp}{dt} = F = \frac{1}{c}\frac{dU}{dt} = \frac{1}{c}Power$$

This is the force pushing the astronaut towards the space-ship with acceleration "a"

(34.63) $$F = Ma;\ a = \frac{F}{M} = \frac{Power}{cM} = \frac{1000W}{3E8 \cdot 95} = 3.51\cdot 10^{-8}\frac{m}{s^2}$$
$$t = \sqrt{\frac{2x}{a}}$$

It will take 629 minutes to get back to the ship. Note that he will have to carry batteries containing 6.5kW·hours of energy. What kind a battery could that be?

I have a 7.4V Lithium battery rated at 1500mAhrs which has a mass of 85 grams. How many such batteries would be required to power the laser above?

$$energy = power \cdot t = I \cdot t \cdot \Delta V = 1.5A \cdot hrs \cdot 7.4V = 40kJ$$

$$\frac{23.4 \cdot MJ}{40kJ} = 586$$

$$586 \cdot 85g = 50kg$$

He would require 586 such batteries having a mass of 50 kg. This means he would be much better off just throwing his battery pack away. Also, it looks like our astronaut with a total mass of 95kg including a 50kg battery pack must be a woman or a dwarf. If she can push her battery pack away from her with a speed of 1m/s she will move towards the space ship with the same speed, thus arriving back in safe harbor in a mere 25seconds. Much better, than the 6.5 hours.

Summary: (34.64)

Intensity of an em wave: $S_{avg} = uc$

Power of an em wave: $S_{avg} A = ucA$

Pressure of an em wave: $\frac{F}{A} = u = \frac{S_{avg}}{c} \Rightarrow F = Au = \frac{S_{avg} A}{c} = \frac{power}{c}$

Momentum p of an em wave in striking a surface A: $p = \frac{U}{c} = \frac{uV}{c}$

Addendum: (optional)
34.8 Mathematical approach to the continuity equation:
$\vec{S}$ **Energy intensity of an electromagnetic wave.**

We show later (see derivation (34.73)) that energy conservation for the e.m. wave in the vacuum can be written as:

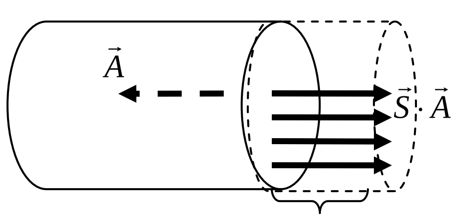
(34.65) $$div\vec{S} = -\frac{\partial u}{\partial t}$$

We accept this formula for the time being without proof. If we apply Gauss's mathematical theorem to this equation it says that the flow of energy through the walls of the surface of a volume created during the time t is equal to the amount of e.m. energy **leaving** the volume during this same time t. Thus, (34.65) is a mathematical statement of energy conservation for electromagnetic waves.

[It is a specialized case of the more general formula which simply says that the total energy (or mass) in a certain volume, can only change if energy (or mass) flows out or into this volume. $div\cdot\vec{j} = -\frac{\partial\rho}{\partial t}; \vec{j} = \rho\vec{v}$]

The net rate of outflow of electromagnetic energy is equal to the loss of electromagnetic energy per unit time.
If we assume a volume in the form of a cylinder whose central axis is parallel to S, and whose length is *c·dt* then, during the time *dt* the Poynting vector (the electro-magnetic wave front, travelling at the speed of light) moves through this whole volume , "carving out a volume" $Adx = Avdt = Acdt$. This is the volume which we use in applying Gauss's law:

(34.66) $$\iiint_V div\vec{S}dV = \oiint_{\partial V}\vec{S}\cdot d\vec{A} = -\iiint_V \frac{\partial}{\partial t}udV = -u\frac{d}{dt}\iiint_V dV = -u\frac{d}{dt}(Ax) = -uAc$$
$$\vec{S}\vec{A} = -SA = -u\frac{d}{dt}(Ax) = -uAc$$



The amount of the electromagnetic energy flux occurring during the time t, necessary for the electromagnetic wave to traverse the volume at the speed of light is equal to the amount of e.m. energy leaving the volume **during the same time *t***.

Volume V=Ac·t

As the electromagnetic wave moves at the speed of light, the volume carved out in the time *t* is Act. During this time t , the Poynting vector $\vec{S}$ travels through the distance ct. The total energy contained in the traversed volume is *uAct*. The total energy decrease per unit time is equal to the rate of outflow of energy expressed by the flux of the Poynting vector $\oiint_{\partial V} \vec{S} \cdot d\vec{A} = -\iiint_V \frac{\partial}{\partial t} u dV = -u \frac{d}{dt} \iiint_V dV = -u \frac{d}{dt}(Ax) = -uAc$

A positive outflow (positive flux) implies negative rate of change in energy contained in the volume enclosed by the surface A. The magnitudes of the two quantities are the same. Thus, we have $SA = ucA$, and after dividing by A we get for the relationship between the magnitudes of S and u: (34.67)

$$\boxed{u = \frac{S}{c}}$$

34.8a General Proof of the continuity equation for an e.m. wave:

$div\vec{S} = -\frac{\partial u}{\partial t}$ *for* $\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0}$.

The total instantaneous energy density associated with the electro-magnetic field is:

(34.68) $$u(t) = u_B(t) + u_E(t) = \frac{1}{2\mu_0}B^2 + \frac{1}{2}\varepsilon_0 E^2 = \varepsilon_0 E^2 = \frac{1}{\mu_0}B^2$$

This must be consistent with the electromagnetic energy carried in any e.m.-wave. We have defined the Poynting vector which gives us the rate of flow of energy in an electromagnetic wave:

(34.69) $$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} \text{ which is the intensity of a wave}$$
$$[S] = \frac{\text{power}}{\text{cross-sectional area A}} = \frac{Watts}{m^2}$$

We need to figure out how the div operator works on the cross-product $\vec{E}\times\vec{B}$ where the electric and magnetic fields satisfy Maxwell's equations in the vacuum (34.2):

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right);\ div\vec{S} = \vec{\nabla}\bullet\vec{S}$$

$$\mu_0 div\vec{S} = \vec{\nabla}\left(\vec{E}\times\vec{B}\right) = \vec{\nabla}\left(\underbrace{\hat{\vec{E}}}_{\uparrow}\times\vec{B}\right) + \vec{\nabla}\left(\vec{E}\times\underbrace{\hat{\vec{B}}}_{\uparrow}\right) = \vec{B}\left(\vec{\nabla}\times\vec{E}\right) - \vec{E}\left(\vec{\nabla}\times\vec{B}\right)$$

(34.70) where we have used the product rule for derivative operators;

we must be careful with the proper order of the vectors in the cross product.

$\underbrace{\hat{\vec{E}}}_{\uparrow}$ means that ∇ operates on this vector only.

Next we use cyclical permutation for this mixed product, so that the vector operator appears directly in front of the vector field it operates on:

We use cyclical permutation $\vec{A}\left(\vec{B}\times\vec{C}\right) = \vec{C}\left(\vec{A}\times\vec{B}\right) = \vec{B}\left(\vec{C}\times\vec{A}\right)$ for

(34.71) $$\vec{\nabla}\left(\underbrace{\hat{\vec{E}}}_{\uparrow}\times\vec{B}\right) = \vec{B}\underbrace{\left(\vec{\nabla}\times\vec{E}\right)}_{-\frac{\partial\vec{B}}{\partial t}} = \vec{B}\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{1}{2}\frac{\partial}{\partial t}\vec{B}^2$$

We do the same for the second term in (34.70)

(34.72) $$\vec{\nabla}\left(\vec{E}\times\underbrace{\hat{\vec{B}}}_{\uparrow}\right) = \underbrace{\hat{\vec{B}}}_{\uparrow}\left(\vec{\nabla}\times\vec{E}\right) = \vec{E}\left(\underbrace{\hat{\vec{B}}}_{\uparrow}\times\vec{\nabla}\right) = -\vec{E}\underbrace{\left(\vec{\nabla}\times\underbrace{\hat{\vec{B}}}_{\uparrow}\right)}_{\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}} = -\vec{E}\left(\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}\right) = \frac{-1}{2}\varepsilon_0\mu_0\frac{\partial\vec{E}^2}{\partial t}$$

For the whole expression we get finally:

(34.73) $$div\vec{S} = -\frac{1}{2\mu_0}\frac{\partial \vec{B}^2}{\partial t} - \frac{1}{2\mu_0 c^2}\frac{\partial \vec{E}^2}{\partial t} = -\frac{1}{2\mu_0}\frac{\partial \vec{B}^2}{\partial t} - \frac{\varepsilon_0}{2}\frac{\partial \vec{E}^2}{\partial t} = -\frac{\partial}{\partial t}\left(u_B + u_E\right)$$

(34.74) $$div\vec{S} = -\frac{1}{2\mu_0}\frac{\partial}{\partial t}(B^2 + \frac{c^2 B^2}{c^2}) = -\frac{1}{\mu_0}\frac{\partial}{\partial t}\vec{B}^2 \equiv -\frac{\partial}{\partial t}u$$

Proof that $\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right) = \vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right) - \vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right)$, which is a mixed product that can be calculated directly, bearing in mind the product rule for derivatives:

(34.75) $$\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \left\langle E_y B_z - B_z E_y, E_z B_x - E_x B_z, E_x B_y - E_y B_x \right\rangle$$
$$= \frac{\partial E_y}{\partial x}B_z + \frac{\partial B_z}{\partial x}E_y - \frac{\partial B_z}{\partial x}E_y - \frac{\partial E_y}{\partial x}B_z +$$
$$+\frac{\partial E_z}{\partial y}B_x + \frac{\partial B_x}{\partial y}E_z - \frac{\partial E_x}{\partial y}B_z - \frac{\partial B_z}{\partial y}E_x +$$
$$\frac{\partial E_x}{\partial z}B_y + \frac{\partial B_y}{\partial z}E_x - \frac{\partial E_y}{\partial z}B_x - \frac{\partial B_x}{\partial z}E_y = \vec{B}\left(\vec{\nabla}\times\vec{E}\right) - \vec{E}\left(\vec{\nabla}\times\vec{B}\right)$$

If we want to write the mixed product in terms of a determinant we have again to pay attention to the fact that we are dealing with derivates which require the product rule if applied to a product of functions:

(34.76) $$\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{E_x} & \mathrm{E_y} & \mathrm{E_z} \\ B_x & B_y & B_z \end{vmatrix} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{B_x} & \mathrm{B_y} & \mathrm{B_z} \\ E_x & E_y & E_z \end{vmatrix}$$

The derivate operators act only on the second row of the determinant.

34.9 Limitations of Maxwell's equations.
Maxwell's equations explain much of the properties of electromagnetic phenomena. However, there are some insurmountable problems with it. For example, according to Maxwell's equations accelerated charges form e.m. waves which carry energy away. Electrons in atoms move in orbits and are therefore being accelerated. They should lose energy and spiral towards the nucleus. This is not the case and can only be resolved through quantum theory. Para-magnetism, dia- and ferro- magnetism also need quantum theory. All the facts about diodes and transistors need quantum theory for their fundamental understanding.
Here are some of the important facts which quantum theory delivers, and which I have occasionally mentioned in my lectures, in addition to the relationships mentioned in the previous section on food for thought.

$$Power = eA\sigma T^4; T \text{ in Kelvin}$$

Black body radiation: (34.77) $e =$ emissivity $0 < e \le 1$; A cross-sectional area;

σ Stefan Boltzmann constant=$5.670 \cdot 10^{-8}$

Particle wave properties:

(34.78)
$$\text{energy of a free particle } E = \hbar\omega = h\nu;$$
$$\text{momemtum of a free particle } p = \hbar k = \frac{h}{\lambda}$$

Heisenberg's Uncertainty Relations:

(34.79)
$$\Delta x \bullet \Delta p_x \ge \frac{\hbar}{2};\ \Delta E \cdot \Delta t \ge \frac{\hbar}{2}$$