Homework: See website.

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32.1 Self-inductance in a circuit.

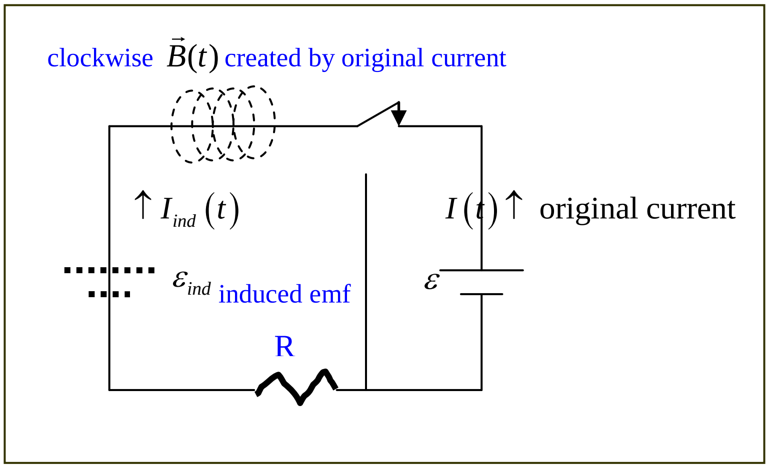
Faraday's law has important consequences for electric circuits. **Every circuit is a loop, and a time changing current will therefore produce a back-emf**. Consider a simple circuit consisting of a resistor R, an emf ε, and a switch. When we turn the switch on the current increases slowly from 0 to a maximum value. The current is obviously varying with time. The magnetic field created by this time varying current is also increasing with time. According **Faraday's law** this magnetic field creates a time varying emf and a time varying induced current in the same loop, which goes against the original current, due to Lenz's law.

(32.1) $$curl\vec{B}(t) = \mu_0 \vec{j}(t) \Rightarrow \oint_{\partial A} \vec{B}(t)\cdot d\vec{s} = \mu_0 I(t)$$

(32.2) $$Bl = \mu_0 I \Rightarrow B(t) = \frac{\mu_0 I(t)}{l}$$

which will induce an emf ε_L in the same circuit according to **Faraday's law**:

(32.3) $$\varepsilon_L(t) = -\underbrace{\frac{d\Phi(t)_B}{dt}}_{\text{rate of magnetic flux through the loop of the circuit}} = -\frac{d}{dt}\frac{\mu_0 I(t)}{l}A = -\underbrace{\frac{\mu_0 A}{l}}_{\text{inductance L}}\frac{dI}{dt}$$



For simplification and demonstration purposes I have multiplied B on the left side in (32.2) with the length of the closed path. Therefore, we see easily that the magnetic field (32.2) in this process of **self-induction** is proportional to the original time-varying current.
It is this same magnetic field which enters **Faraday's law** on the right side of (32.3)

We see that the emf of **self-inductance** is proportional to the rate of change of the original current.
We introduce the new quantity **L, the inductance of the circuit**, as the **proportionality factor between the emf and the rate of change of the original current**. For a coil with N tightly wound loops the flux through N loops is equal to N times the flux through a single loop. An emf is created in each single loop, and the total emf is the sum of all the emfs. There is only one single current, therefore:

(32.4) $$\boxed{\varepsilon_L = -N\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}}$$

The **self induced emf** ε_L is always proportional to the **rate of change of the original current**.
(Note that from now on we avoid the letters "l" or "L" for lengths. We reserve the letter L for self-inductance.)

From (32.4) we get:

(32.5) $$Nd\Phi_B = LdI$$

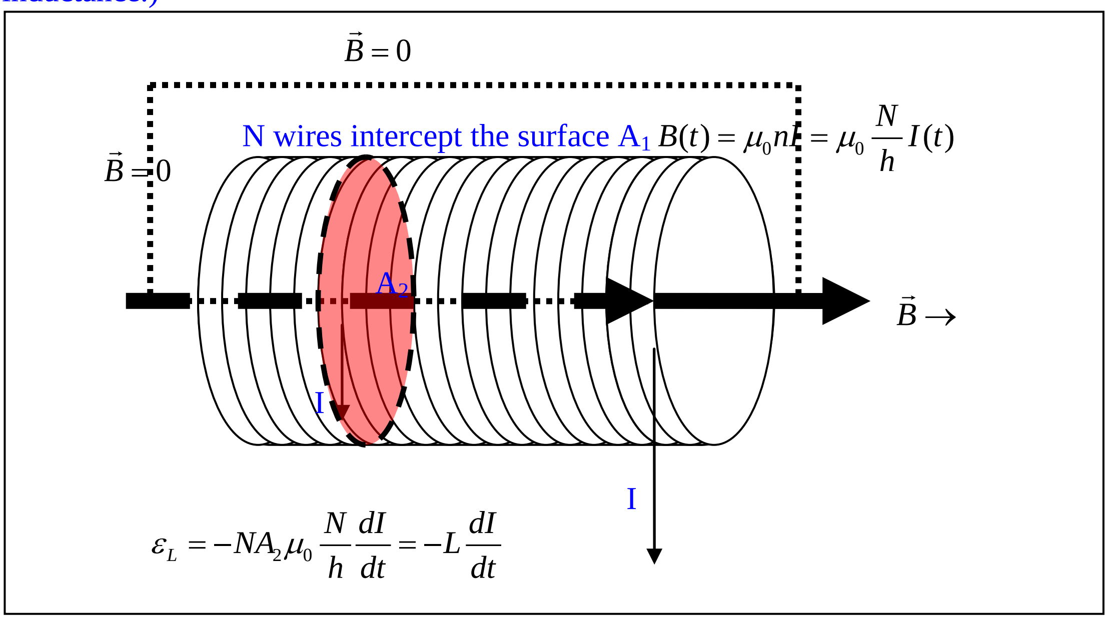
The change in flux is directly proportional to the change in current.When there is no current there is no flux, and vice versa. Integrating, we get

(32.6) $$LdI = Nd\Phi_B \Rightarrow L = \frac{N\Phi_B}{I}$$

(32.7) $$\boxed{L = \frac{N\Phi_B}{I}}$$

To summarize: the time varying current I(t) in a circuit creates a time varying magnetic field B(t) in the circuit (Ampere's law). The time varying magnetic field induces a time-varying back-emf (Faraday's law) and an induced current in the circuit.
Both, the self induced emf ε_L and the induced current I_{ind} are directed against the original emf ε and the original current I that created them.

Example 32.1: Find the **inductance of a uniformly wound very long solenoid** having N turns and length h. (We use the letter h for length instead of L which is now reserved for the inductance.)



From Ampere's law we know: $curl\vec{B} = \mu_0 \vec{j} \Rightarrow Bh = \mu_0 NI$

Recall that in the calculation of the magnetic field the Ampereian surface for which we calculated the flux of the current density was a rectangular surface with its long side parallel to the magnetic field inside the solenoid. In the figure above we call this surface A_1.

(32.8) $$B(t) = \mu_0 n I(t) = \mu_0 \frac{N}{h} I(t)$$

In order to calculate the emf induced by the time changing magnetic field we **calculate the flux of this magnetic field through N surfaces created by the N turns of wire**. Each of these surfaces is perpendicular to the Ampereian surface. We should call them Faraday surfaces. The circulation around each such loop creates a back-emf, which is equal to the time changing magnetic flux through this loop. $\varepsilon_i = -\frac{d\Phi_B}{dt}$. For N loops we get a total back-emf:

(32.9) $$\varepsilon = \sum_{i=1}^{N} \varepsilon_i = -N\frac{d\Phi_B}{dt}$$

The magnetic flux for a single loop of wire is equal to the magnetic field inside the solenoid times the surface of one single loop of wire or $A_2 B$. Therefore,

(32.10) $$L = \frac{N\Phi_B}{I} = \frac{NA_2 B}{I} = \frac{NA_2}{I}\mu_0 \frac{N}{h} I = \frac{\mu_0 N^2 A_2}{h} = \mu_0 n^2 \underbrace{A_2 h}_{\text{Volume of the interior solenoid.}} = \mu_0 n^2 V$$

(32.11) $$\boxed{L_{solenoid} = \mu_0 n^2 V}$$

By introducing a ferromagnetic material into the solenoid the self-inductance can be greatly increased, because it strengthens the magnetic field. This effect gets absorbed in the value for μ_0 which then is **not** the permeability of empty space, and is called μ without the 0..

Let us calculate the inductance L for the solenoid directly from Faraday's law:

$$\iint_{A_2} curl\vec{E} \cdot d\vec{A}_2 = \oint_{\partial A_2} \vec{E} \cdot d\vec{s} = -\frac{d}{dt}\iint_{A_2} \vec{B} \bullet d\vec{A} = -NA_2 \frac{dB}{dt} = -NA_2 \mu_0 \frac{N}{h}\frac{dI}{dt}$$

The line integral of the induced electric field must be taken N times for each loop. For a tightly wound solenoid it is equal to

(32.12) $$\varepsilon_L = \sum \varepsilon_i = N\oint_{\partial A_2} \vec{E} \cdot d\vec{s} = 2\pi r N E = -N\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$$

The flux-integral yields N times the magnetic flux through a simple surface, which is the circular loop created by a single circular loop of wire:

(32.13) $$-\frac{d}{dt}\iint_{A_2} \vec{B} \bullet d\vec{A} = -NA_2 \frac{dB}{dt} = -NA_2 \mu_0 \frac{N}{h}\frac{dI}{dt}$$

With the definition of

(32.14) $$\varepsilon_L = -L\frac{dI}{dt}$$

we get the result in (32.11):

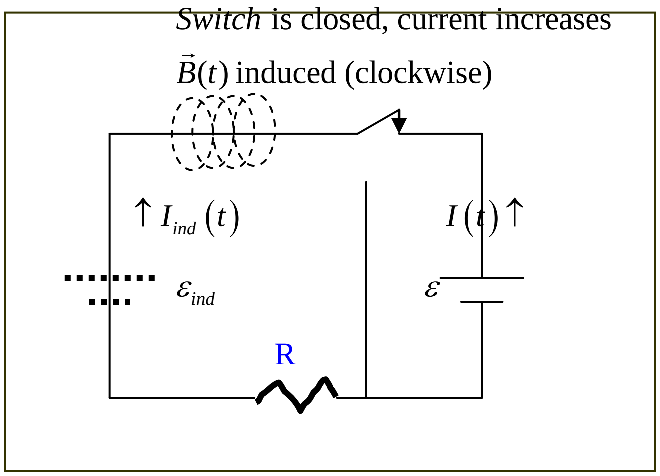
(32.15) $$\varepsilon_L = -L\frac{dI}{dt} = -NA_2 \mu_0 \frac{N}{h}\frac{dI}{dt} \Rightarrow L = \mu_0 N^2 \frac{A_2}{h} = \mu_0 \frac{N}{h}\frac{N}{h} h A_2 = \mu_0 n^2 V$$

32.2 RL-Circuits.

Solving the equations of an RL dc circuit during start-up and shut-down:

Let us further study what happens in a dc-circuit with a resistor R. (All circuits have a resistance, whether we explicitly include a resistor or not.)

32.2a Charging the capacitor:
When we close the switch in such a circuit, allowing a current to flow, the current will not immediately flow with its maximum value. It is growing from 0 to its maximum value ε/R. This means of course that the current changes with time, which again means that during this time a back-emf is established in the circuit according to Faraday's law: $\varepsilon_L = -L\frac{dI}{dt}$. The emf acts against the original current according to Lenz's law. When, after the current has reached its maximum value, we remove the original power source ε, the current will decrease and again cause a back emf, this time to oppose the current's decrease.



Just like the resistance R was a measure of the opposition to the current $\Delta V = RI$ so is the inductance L a **measure of the resistance to the change in the original current** $\varepsilon_L = -L\frac{dI}{dt}$. Both quantities ΔV=-RI and $\varepsilon_L = -L\frac{dI}{dt}$ (energies per unit charge) will reduce the voltage supplied by the original outside power supply. We still have energy conservation when we complete the loop of a circuit. The power supplied by the outside source ε equals the power used by the circuit.

When the switch is closed in the circuit above with a power supply ε the current in the circuit increases from 0 to its maximum value $\frac{\varepsilon}{R}$.
We apply energy conservation to the whole loop and get a familiar differential equation:
We need to solve our equation for I(t). Let us first do this through simple separation of the variables:

$$\varepsilon - RI - L\frac{dI}{dt} = 0 \Rightarrow -L\frac{dI}{dt} = RI - \varepsilon$$

(32.16)
$$\frac{dI}{RI-\varepsilon} = -\frac{dt}{L} \Rightarrow \frac{dI}{I-\frac{\varepsilon}{R}} = -\frac{R}{L}dt$$

We get the solution of an exponentially increasing current with the time-constant $\tau = \frac{L}{R}$

$$I(t) = \frac{\varepsilon}{R}\left(1-e^{-\frac{R}{L}\cdot t}\right) \Rightarrow \frac{dI}{dt} = \frac{I_0}{\tau}e^{-\frac{t}{\tau}} > 0$$

After a time of 4.6τ the current reaches 99% of its maximum value of $\frac{\varepsilon}{R}$. Let us check this:

(32.17)
$$0.99\frac{\varepsilon}{R}=\frac{\varepsilon}{R}\left(1-e^{-\frac{R}{L}\cdot t}\right)\Rightarrow 0.99-1=-e^{-\frac{R}{L}\cdot t}\Rightarrow 0.01=e^{-\frac{R}{L}\cdot t}$$
$$\ln 100=\frac{R}{L}t\Rightarrow t=4.6\tau$$

During this time we have an **increasing** current $I(t)$, therefore also a time varying magnetic field, therefore a self-induced emf. $\varepsilon_{ind}=-L\frac{dI}{dt}$ Therefore, the polarity of the induced emf is opposite to the original emf. While the induced current is flowing, power is being delivered to the magnetic field which builds up in the coil because of $P=\varepsilon_L I=L\frac{dI}{dt}I\Rightarrow U_B=\frac{1}{2}LI^2$. At the same time power is being dissipated in the resistor as heat according to $P=RI^2$.

32.2b Different method to solve the differential equation (32.16):

(32.18)
$$L\frac{dI}{dt}=-RI+\varepsilon$$

We first recognize that the d.e. contains a constant. The general rule for solving d.e.'s says to first solve the d.e. without the constant and then to add a special solution which satisfies the equation with the constant. The d.e. without the constant is easily solved:

(32.19)
$$L\frac{dI}{dt}=-RI\Rightarrow I(t)=Ae^{-\frac{R}{L}t}$$

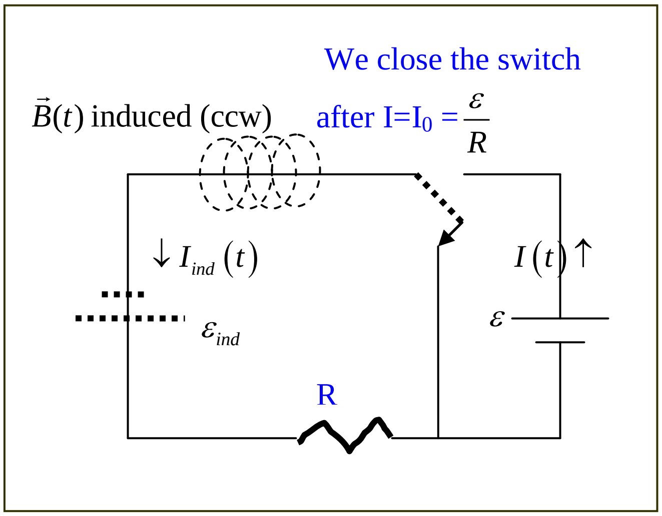
A **special constant solution** is given by $I=\frac{\varepsilon}{R}$. You can verify that by putting that solution into the original d.e. $0=-R\frac{\varepsilon}{R}+\varepsilon$ Our general solution is therefore obtained by **adding the special solution to the general solution** to yield:

(32.20)
$$I=Ae^{-\frac{R}{L}t}+\frac{\varepsilon}{R}$$

Now, we apply the initial condition that the current is 0 at t=0 and get:

(32.21)
$$0=A+\frac{\varepsilon}{R}\Rightarrow A=-\frac{\varepsilon}{R}$$

This gives as the same solution we obtained before:(32.22) $I(t)=\frac{\varepsilon}{R}\left(1-e^{-\frac{R}{L}\cdot t}\right)$



32.2c Discharging the coil:
When, after a while, we disconnect the power source, the current in the circuit will not go to 0 in an instant but will also decrease exponentially. It is again impeded by the induced emf which this time opposes the decrease of the current. We can see this again by solving the differential equation of the circuit (which has been energized previously) after the power supply has been removed.
The switch in the circuit is now closed. The current now decreases because it dissipates energy in the resistor and no more outside power is being supplied.
$\frac{\partial \vec{B}}{\partial t} < 0$ the polarity is again reversed.

(32.23)
$$RI + L\frac{dI}{dt} = 0 \Rightarrow \frac{dI}{I} = -\frac{L}{R}dt \Rightarrow$$
$$I = I_0 e^{-\frac{t}{\tau}} \Rightarrow \frac{dI}{dt} = -\frac{I_0}{\tau}e^{-\frac{t}{\tau}} < 0$$

(The time constant $\tau = \frac{R}{L}$)

Summing this up, we need to introduce a new self induced emf into any circuit containing a time varying current. This is done by adding an **inductor coil** into the circuit, usually symbolized by a spiral along the circuit, with the value L for its inductance. In this way, the whole analysis gets reduced to writing down Kirchhoff's rules for the circuit as an **instantaneous** equation. **Just like with the voltage drop across a resistor we have an emf drop across such an inductor coil**. As the origin of the voltage ultimately does not matter, we also talk about a voltage drop across an inductor coil:

(32.24)
$$\Delta V_R = -RI \text{ and } \varepsilon_{ind} = \Delta V_L = -L\frac{dI}{dt}$$

32.3 Energy in a Magnetic Field.
We start again with an LR circuit containing an emf. The instantaneous Kirchhoff rule reads like the electric energy conservation. (We used Kirchhoff's laws so far mostly for dc-currents. They are just a convenient summary of the laws of charge and energy conservation, which must hold for any circuit, including those with time varying currents and voltages. This is why we call Kirchhoff's rules now "instantaneous.")

(32.25)
$$\varepsilon - RI - L\frac{dI}{dt} = 0$$

The energy of the emf will go to the resistor, where it is dissipated as heat, and the coil, where it is stored in the magnetic field. To get the rate of change of energy in the circuit, i.e. its power we simply multiply (32.25) with I. ($Power = I\Delta V = I\varepsilon$)

(32.26) $$\varepsilon I = \underbrace{RI^2}_{\text{power delivered to the resistor}} + \underbrace{LI\frac{dI}{dt}}_{\text{power delivered to the inductance}=\frac{dU_B}{dt}}$$

(32.27) $$\text{power delivered to the inductor coil } P_L = \frac{dU_B}{dt} = LI\frac{dI}{dt}$$

Integrating this over time:

(32.28) $$\int dU_B = \int LIdI = \frac{1}{2}LI^2$$

(32.29) $$\boxed{U_B = \frac{1}{2}LI^2}$$

Recall that the energy in the capacitor was equal to

(32.30) $$U_C = \frac{1}{2}C(\Delta V)^2 = \frac{Q^2}{2C}$$

which allowed us to calculate the energy density in the empty space between the plates of a parallel plate capacitor as:

(32.31) $$\boxed{u_E = \frac{1}{2}\varepsilon_0 E^2}$$

For the magnetic energy inside the space of a solenoidal coil we need L from (32.11).

(32.32) $$L = \mu_0 n^2 V$$

For the current in the coils of the solenoid, expressed by the magnetic field we use:

(32.33) $$B = \mu_0 nI \Leftrightarrow I = \frac{B}{\mu_0 n}$$

Inserting L and I into (32.29) gives us :

(32.34) $$U_B = \frac{1}{2}\left(\mu_0 n^2 V\right)\left(\frac{B}{\mu_0 n}\right)^2$$
$$U_B = \frac{B^2}{2\mu_0}V$$

It follows that the **magnetic energy density** $u_B = U_B/V$ inside a coil is given by :

(32.35) $$\boxed{u_B = \frac{B^2}{2\mu_0}}$$

32.3a Energy in a coaxial cable:

In a coaxial cable the inner thin cylindrical cable carries the current from the source and the outer thin cylindrical cable carries the current back to the source, thus closing a loop. The inside of the inner cylindrical cable has no magnetic field (Ampere's law.) On the outside of the outer cable there is also no magnetic field as the total current through a cross-section of the cable is 0. The only magnetic field exists between the two cable surfaces. We have seen earlier that the magnetic field outside of a conduction wire curls around the wire (inner cylinder) and is equal to

(32.36) $$B(r)=\frac{\mu_0 I}{2\pi r} \text{ from Ampere's law: } 2\pi r B=\mu_0 I$$

In order to calculate the energy

(32.37) $$U_B=\frac{1}{2}LI^2$$

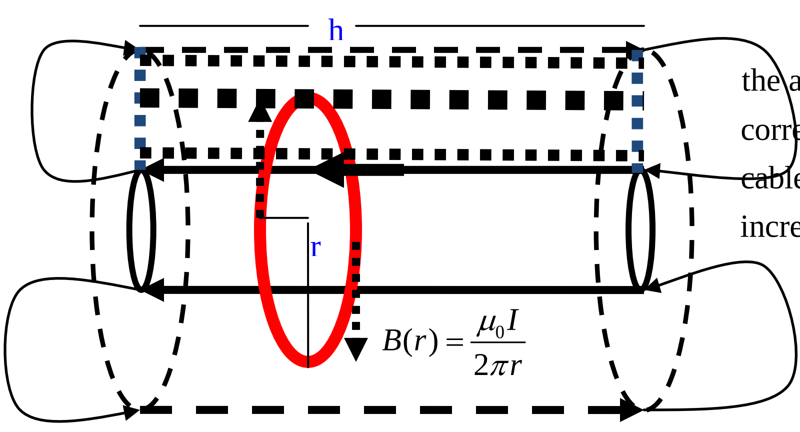
contained in the magnetic field of the coaxial cable we need to find the inductance L of the cable: which is given by :

(32.38) $$\boxed{L=\frac{N\Phi_B}{I}}$$

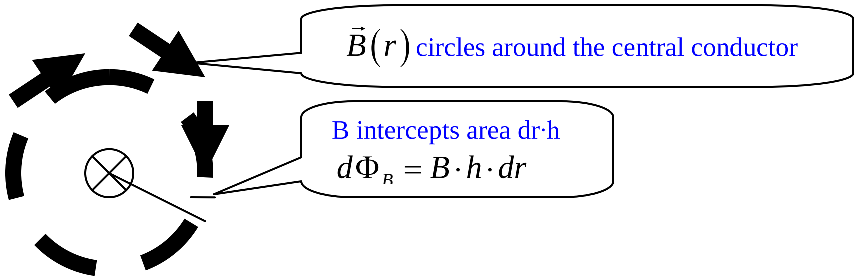
This is in connection with Faraday's law, which uses surfaces perpendiclar to the surface used in determining the magnetic field in Ampere's law. Call them the Faraday surfaces.
We need the surface through which the magnetic field flows and around which a current and emf is being created:

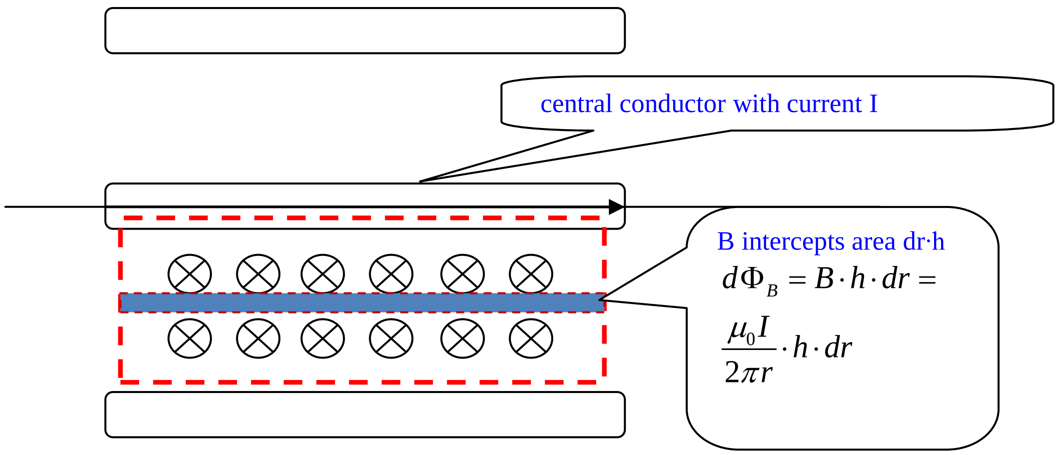
(32.39) $$\oint_{\partial A}\vec{E}\cdot d\vec{s}=-\frac{\partial}{\partial t}\iint_A \vec{B}\bullet d\vec{A}$$

So, we need to calculate the flux of the magnetic field $B(r)=\frac{\mu_0 I}{2\pi r}$ through a rectangular cross-section of the cable (along the cable, not perpendicular to it), in which the magnetic field is tangential to the concentric circle around the inner cable with radius r>a (where a is the inner radius of the inside cable.) (Don't confuse the coaxial cable with a solenoid!) The total Faraday surface is the rectangle of width (b-a) and length h. The magnetic field is perpendicular to that surface and varies with the distance r from the central axis of the concentric cylinders. The small area dA(r) of the cylindrical surface lies at a distance r from the center line and has the thickness dr. Its length is the length of the cable h, thus $dA(r)=hdr$



the area of the flux $d\Phi_B = Bhdr$ corresponds to the height of the cable (length) times the radial increment dr.





We get for the flux:

(32.40) $$\Phi_B = \int_a^b \frac{\mu_0 I}{2\pi r} h dr = \frac{\mu_0 I h}{2\pi} \ln \frac{b}{a}$$

For L we get :

(32.41) $$L = \frac{N\Phi_B}{I} = \frac{N\mu_0 h}{2\pi} \ln \frac{b}{a}; N = 1$$

Fo the total magnetic energy stored in the coaxial cable we get :

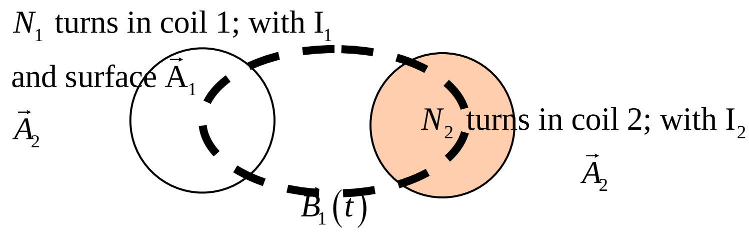
(32.42) $$U_B = \frac{1}{2} L I^2 = \frac{\mu_0 h I^2}{4\pi} \ln \frac{b}{a}$$

32.5 Mutual Inductance:

If we have a time varying current I_1 in circuit 1, and a time varying current I_2 in an adjacent circuit 2, I_1 will induce an emf in circuit 2 and I_2 will induce an emf in circuit 1. We talk about mutual inductance M.

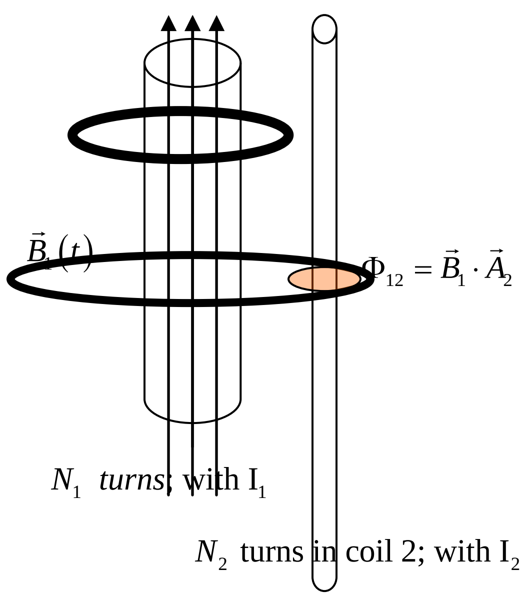
If we have a time-varying magnetic field $\vec{B}_1(t)$ (from a time varying current) then obviously the field creates a flux through any closed surface, not just through the one responsible for its field (which leads to the self inductance L we just discussed.)

If there is another closed loop with a surface close by then, the magetic field will create a flux Φ_{12} and therefore an induced emf ε_2 in that loop also, and then vice versa.



The magnetic field $\vec{B}_1$ created by the left loop intercepts the right loop, creating a magnetic flux Φ_{12} of the magnetic field $\vec{B}_1$ through the surface $\vec{A}_2$.
If we have a time dependent current and magnetic field the change in flux will create an emf ε_2 in the second loop.

$$\varepsilon_2 = -N_2 \frac{d}{dt}\underbrace{\left(\vec{B}_1 \vec{A}_2\right)}_{\Phi_{12}} = -M_{12}\frac{dI_1}{dt}$$



(32.43) which provides the definition of the mutual inductance M_{12}

Obviously, the magnetic field $\vec{B}_2$ created by the current I_2 in the second coil, will create a flux through the first loop $\Phi_{21} = \vec{B}_2\vec{A}_1$ and therefore also an induced emf ε_1 .
We define the mutual inductance accordingly:

$$\varepsilon_1 = -N_1 \frac{d}{dt}\underbrace{\left(\vec{B}_2 \vec{A}_1\right)}_{\Phi_{21}} = -M_{21}\frac{dI_2}{dt}$$

(32.44) which provides the definition of the mutual inductance M_{21}

It turns out that the two mutual inductances are equal and we don't need to distinguish between them.

(32.45) $$M = M_{12} = M_{21}$$

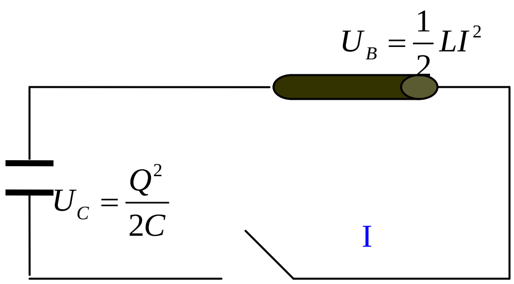
The mutual inductance depends on the shape and make of the coil. Thus, each coil creates a new emf in the other coil:

(32.46) $$\boxed{\varepsilon_1 = -M\frac{dI_2}{dt} \quad and \quad \varepsilon_2 = -M\frac{dI_1}{dt}}$$

The mutual inductance has the same form as the self-inductance. It plays a role in transformers, which we shall discuss later.

<u>32.5 Oscillations in an LC circuit</u>.

Let us consider a simple circuit with a capacitor C, a coil L, and a switch. **<u>The capacitor is originally fully charged and the switch is open</u>**. When we **close the switch** a time-varying current runs through the circuit, which will induce a back emf in the coil. The mathematical expression of this time varying charge is the solution of a differential equation. We bear in mind that the energy of the capacitor decreases as the energy of the coil increases, and vice versa. We can use the energy approach, noticing that the total energy in the circuit remains constant as there is no resistance R in the circuit:



(32.47) $U = \frac{Q^2(t)}{2C} + \frac{1}{2}LI^2(t) = \text{constant}$

The time derivative is 0 because there is no dissipation of energy in a resistor :

(32.48) $\frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = 0 = \frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0$

(after dividing by I=dQ/dt)

This is a differential equation of the second order in Q, just like the d.e. of a spring:

(32.49)

$$\ddot{Q} + \frac{1}{LC}Q = 0 \Leftrightarrow \ddot{x} + \frac{k}{m}x = 0$$

We arrive at the same equation by simply writing down Kirchhoffs rules for a circuit without R and powersupply: $-L\frac{dI}{dt} - \frac{Q}{C} = 0 \Rightarrow L\ddot{Q} + \frac{1}{C}Q = 0 \Rightarrow \ddot{Q} + \frac{1}{LC}Q = 0$

<u>32.5a Solving d.e. by using complex functions</u>:

This is a good time to review how we solve such equations with complex numbers. You can check out the paper <u>ch32Complex Oscillations</u> on the website (Do it, you will need it!). We will make extensive us of this in the following two chapters. It is not covered in your text book.

Use the complex trial solution

(32.50) $$\hat{Q}(t) = Q_0 e^{i\alpha t} = Q_0(\cos\omega t + i\sin\omega t)$$

Use the fact that the derivative of an exponential function turns into a multiplication :

(32.51)
$$\frac{d\hat{Q}(t)}{dt} = \frac{d}{dt}Q_0 e^{i\alpha t} = i\alpha\hat{Q}(t) = \dot{Q}$$
$$\frac{d^2\hat{Q}(t)}{dt^2} = \frac{d}{dt}Q_0 e^{i\alpha t} = (i\alpha)^2\hat{Q}(t) = -\alpha^2\hat{Q}$$

With this equation (32.49) becomes :

(32.52) $$\left(-\alpha^2 + \omega_0^2\right)\hat{Q}(t) = 0 \Rightarrow \alpha = \pm\omega_0$$

Using the initial condition that Q(t=0)=Q_0 we get the familiar solution to (32.49):

(32.53) $$\boxed{Q = Q_0 \cos\omega_0 t;\ \text{with}\ \omega_0^{\ 2} = \frac{1}{LC}}$$

We could have also just written down Kirchhoffs rules for this circuit :

(32.54) $$-L\frac{dI}{dt} - \frac{Q}{C} = 0 \Rightarrow L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0 \Rightarrow$$
$$\ddot{Q} + \frac{1}{LC}Q = 0 \Rightarrow Q(t) = Q_0\cos(\omega_0 t + \varphi)\ \text{with}\ \omega_0^2 = \frac{1}{LC}$$

Compare these equations and solutions with the equations for the oscillations of a spring :

(32.55) $$\ddot{x} + \frac{k}{m}x = \ddot{x} + \omega_0^2 x = 0 \Rightarrow x = x_0\cos\left(\omega_0 t + \varphi\right)$$

The potential energy of the spring corresponds to

(32.56) $$\frac{1}{2}kx^2 \Leftrightarrow \frac{Q^2}{2C}$$

The kinetic energy of the spring corresponds to the energy of the coil :

(32.57) $$\frac{1}{2}m\text{v}^2 = \frac{1}{2}m\dot{x}^2 \Leftrightarrow= \frac{1}{2}L\dot{Q}^2 = \frac{1}{2}LI^2$$

32.6 The RLC circuit (without exterior power source).

If we **introduce a resistor R** ($\Delta V = -RI$) into the circuit above (no emf) we get the situation analogous to that of the spring with a damping factor b ($F = -bv$). The circuit now loses energy according to $-RI^2$, just like the spring loses energy according to $-bv^2$. Starting with a fully charged capacitor, corresponds to starting the spring with its maximum potential energy.
By using the same process as before we get the d.e.

(32.58) $$\not{\varepsilon} - RI - L\frac{dI}{dt} - \frac{Q}{C} = 0 \Rightarrow L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

(32.59) $$\ddot{Q} + \frac{R}{L}\dot{Q} + \frac{1}{LC}Q = 0$$

Use the trial solution: $Q = Q_0 e^{i\alpha t} \Rightarrow \dot{Q} = i\alpha Q;\ and\ \ddot{Q} = -\alpha^2 Q$
This leads to the quadratic equation in α (Learn how to do this!):

(32.60) $$-\alpha^2 + i\alpha\frac{R}{L} + \omega_0^2 = 0$$

(32.61) $$\alpha = \frac{i\frac{R}{L} \pm \sqrt{-\frac{R^2}{L^2} + 4\omega_0^2}}{2} = i\frac{R}{2L} \pm \underbrace{\sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}}_{\omega_1}$$

(32.62) $$Q(t) = Q_0 e^{i\alpha t} = Q_0 e^{-\frac{R}{2L}t \pm i\omega_1 t} = Q_0 e^{-\frac{R}{2L}t}\left(\cos\omega_1 t \pm \sin\omega_1 t\right)$$

(32.63)
$$\boxed{\begin{gathered}L\frac{dI}{dt}+RI+\frac{Q}{C}=0\Rightarrow\\ \ddot{Q}+\underbrace{\frac{R}{L}}_{b/m}\dot{Q}+\underbrace{\frac{1}{LC}}_{\omega_0^2}Q=0\end{gathered}}$$

This equation corresponds to the spring equation:

(32.64)
$$\ddot{x}+\frac{b}{m}\dot{x}+\frac{k}{m}x=0$$

Obviously, we get the same solution which is a damped oscillation :

(32.65)
$$x=x_0e^{-\frac{b}{2m}t}\cos\omega_1 t;\omega_1=\sqrt{\omega_0^2-\left(\frac{b}{2m}\right)^2};\omega_0^2=\frac{k}{m}$$

Comparing the two differential equations we just need to establish a correspondents between the constants:

(32.66)
$$\begin{gathered}b\to R\\ m\to L\\ \frac{k}{m}=\omega_0{}^2=\frac{1}{LC}\end{gathered}$$

Thus :

(32.67)
$$Q(t)=Q_0e^{-\frac{R}{2L}t}\cos\omega_1 t;\omega_1=\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2}$$

Energy is lost by this circuit to the tune of $-b\mathrm{v}^2$ in terms of the spring's constants. This translates to the LC circuit as:

(32.68)
$$-R\dot{Q}^2=-RI^2$$

just as we expected.

Let us double check this by starting with the instantaneous energy of a circuit which is given by:(32.47) $U=\frac{Q^2}{2C}+\frac{1}{2}LI^2$

In the case of the LC circuit this energy is constant. In the case of the LRC circuit we know that energy is lost in the resistor:

(32.69)
$$\frac{dU}{dt}=\frac{d}{dt}\left(\frac{Q^2}{2C}+\frac{1}{2}LI^2\right)=\frac{1}{2C}2Q\underbrace{\dot{Q}}_{I}+\frac{L}{2}2I\dot{I}=\frac{Q}{C}I+LI\dot{I}=I\left(\frac{Q}{C}+L\dot{I}\right)$$

Looking at the original differential equation $-RI-L\dot{I}-\frac{Q}{C}=0$ we see that the term in parentheses is equal to –RI. Therefore, the change in total energy of the circuit is equal to the energy loss to heat or internal energy:

(32.70)
$$\boxed{\frac{dU}{dt}=I\underbrace{\left(\frac{Q}{C}+L\dot{I}\right)}_{-RI}=-RI^2}$$