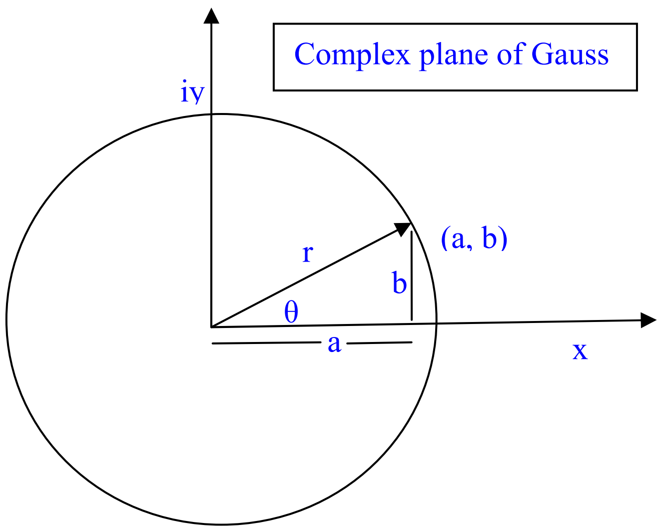
Carl Friedrich Gauss introduced complex numbers: In the complex plane the y-axis is the imaginary axis and the x-axis is the real axis. Any complex number z can then be written in terms of its real part plus its imaginary part:

$$\hat{z} = x + iy \Rightarrow$$

$$\hat{z} = r\left(\cos\theta + i\sin\theta\right) = re^{i\theta} \text{ Euler formula} \tag{1.1}$$

One can easily prove Euler's formula by expanding the exponential function e^x in a McLaurin power series, and then substituting x with iθ. We must just use that i^2=-1; i^3=-i; i^4=1; i^5=i



Complex numbers help us to find all solutions of an algebraic equation like, for example: x^3 =-1.

All we have to do is write our number, real, or complex, in terms of an exponential: (1.2) $\hat{z}^3 = -1 = e^{i\pi + 2ni\pi}$ for n=0,1,2,3... then we take the third root of this equation and take all non repetitive solutions into account.

$$a)\left(e^{i(\pi+2n\pi)}\right)^{\frac{1}{3}} = e^{i\frac{(\pi+2n\pi)}{3}} \Rightarrow \hat{z}_0 = e^{i\frac{(\pi+2\pi 0)}{3}} = e^{i\frac{\pi}{3}}; n=0$$

$$b)\hat{z}_1 = e^{i\frac{(\pi+2\pi)}{3}} = e^{i\pi}; n=1$$

$c)\hat{z}_2 = e^{i\frac{(\pi+4\pi)}{3}} = e^{i\frac{5\pi}{3}}; n=2$ we have three unique solutions, which can of course also be written in trigonometric and algebraic formats.

The same can be done for any equation of the form

$a) a + ib = \hat{z}^n$

First, write a+ib in terms of an exponential function $re^{i(\theta+2k\pi)}; k = 0,1,2...$

(1.3) $b) r = \sqrt{a^2+b^2} = |a+ib| = \sqrt{(a+ib)\cdot(a-ib)}$

$c) \theta = \arctan\frac{b}{a}$

(1.4)
$$\hat{z}^n = a+ib = \sqrt{a^2+b^2}\,e^{i(\theta+2k\pi)} \quad \theta = \tan^{-1}\frac{b}{a} \text{ in radians}$$
$$\hat{z}_k = r^{\frac{1}{n}} e^{i\frac{(\theta+2k\pi)}{n}} \; ; k = 0,1,2,3...n-1$$

(1.5)

$$example: 3-4i = \hat{z}^3; r = \sqrt{3^2+4^2} = 5; \theta = \arctan\frac{-4}{3} = -0.92795218 \approx -0.93$$

$$\hat{z}_k = 5^{\frac{1}{3}} e^{i\frac{(-0.93+2k\pi)}{3}} \; ; k = 0,1,2$$

$$\hat{z}_0 = 5^{\frac{1}{3}} e^{i\frac{(-0.93)}{3}} = 1.71e^{-i0.31} = 1.71(\cos 0.31 - i\sin 0.31) = 1.71(0.923 - i0.305) = 1.578 - i0.5217$$

$$\hat{z}_1 = 5^{\frac{1}{3}} e^{i\frac{(-0.93+2\pi)}{3}} = 1.71e^{i1.7844} = 1.71(-0.212 + i0.9773) = -0.3625 + i1.6711$$

$$\hat{z}_2 = 5^{\frac{1}{3}} e^{i\frac{(-0.93+4\pi)}{3}} = 1.71e^{i3.8788} = 1.71(-0.74035 - i0.6722) = -1.266 - i1.150$$

The **complex conjugate** z^* (z-star) number of a complex number z is obtained by turning all signs of the imaginary unit into their opposite: +i becomes –i and vice versa.

(1.6)
$\hat{z} = a+ib = re^{i\theta}; \hat{z}^* = a-ib = re^{-i\theta};$

We define the norm (or magnitude) of a complex number as:

$|\hat{z}| = \sqrt{\hat{z}\hat{z}^*} = \sqrt{a^2+b^2} = r$

Rationalization of the denominator:

(1.7)
$$a) \frac{1}{a+ib} = \frac{\hat{z}^*}{\hat{z}\hat{z}^*} = \frac{a-ib}{a^2+b^2} = re^{-i\beta}; \; r = \frac{\sqrt{a^2+b^2}}{a^2+b^2} = \frac{1}{\sqrt{a^2+b^2}}; \beta = \tan^{-1}\frac{b}{a}$$
$$b) \frac{1}{a+ib} = \frac{1}{\sqrt{a^2+b^2}} e^{-i\beta}$$

Some interesting properties are a consequence of Euler's formula (1.1):

(1.8)
$$(\cos\theta + i\sin\theta)^n = \sum_{k=0}^{n} \binom{n}{k} (\cos\theta)^k (i\sin\theta)^{n-k} = \cos n\theta + i\sin n\theta \; DeMoivre$$
$$e^{i\pi n} = \sum_{k=0}^{n} \binom{n}{k} (\cos\pi)^k (i\sin\pi)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (0)^{n-k} = \binom{n}{n} (-1)^n (0)^{n-n} = (-1)^n$$

$$a)(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta = \cos 2\theta + i\sin 2\theta$$

$$b)\text{binomial coefficients: } \binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!} \text{ for k}\rangle\text{0 and 0 for k=0}$$

$$\binom{2}{0} = 1; \binom{2}{1} = 2; \binom{2}{2} = 1; \binom{n}{0} = 1 = \binom{n}{n}$$

$$c)(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

$$\Rightarrow \cos^3\theta - 3\cos\theta\sin^2\theta = \cos 3\theta; 3\cos^2\theta\sin\theta - \sin^3\theta = \sin 3\theta$$

Recall the power expansion of a function into a McLaurin series:

(1.9)
$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(0)\frac{x^k}{k!}; \text{ with } f^{(k)}(0) \equiv \frac{d^k f(x)}{dx^k}\Big|x=0$$

which is the special case of the Taylor series, expansion around the point a:

(1.10)
$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(a)\frac{(x-a)^k}{k!}; \text{ with } f^{(k)}(a) \equiv \frac{d^k f(x)}{dx^k}\Big|x=a$$

The expansion for e^x is then:

(1.11)
$$a)e^x = 1 + x + \frac{x^2}{2!}\cdots + \frac{x^n}{n!} + \cdots \text{ and}$$
$$b)e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \frac{\theta^8}{8!}\cdots = \cos\theta + i\sin\theta \Rightarrow$$
$$c)e^{-i\theta} = \cos\theta - i\sin\theta \Rightarrow$$
$$d)\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

One of the interesting attributes of complex exponential functions is their derivatives. A derivate is simply reduced to a multiplication by iω, and an integration is reduced to a division by iω:

(1.12)
$$\hat{f}(t) = x_0 e^{i\omega t} \Rightarrow \frac{d\hat{f}(t)}{dt} = \frac{dx_0 e^{i\omega t}}{dt} = i\omega x_0 e^{i\omega t} = i\omega \hat{f}(t) \Rightarrow$$
$$\frac{d^n \hat{f}(t)}{dt^n} = (i\omega)^n \hat{f}(t)$$

All of this becomes very useful when we want to solve certain differential equations, as they occur in physics. Consider, for example, the differential equation of a spring:

Simple Harmonic Motion (SHM):

Note that k is the spring constant here, not the wave number.

$F = -kx$; k is the spring constant!

Newton's second law applies:

(1.13)
$$F = ma \equiv m\ddot{x} \equiv m\frac{d^2x}{dt^2} = -kx$$
$$\ddot{x} + \frac{k}{m}x = 0$$ this is a linear differential equation of the second order.

It has two independent solutions, which combine to form the general solution of the d.e.

The d.e. requires two initial conditions to integrate and to form a unique solution.

We see that cosωt, sinωt, and $e^{i\omega t}$, $e^{-i\omega t}$ are solutions.

$Let\ x = A\cos\omega t \Rightarrow \ddot{x} = -\omega^2 x$ therefore this is a solution if

(1.14)
$$\omega^2 = \frac{k}{m}$$
$$x = A\sin\omega t \text{ and } A\sin\underbrace{(\omega t + \varphi)}_{\text{phase}};$$ φ is called a constant phase shift;

these are all functions which satisfy the differential equation.

Any linear combination of such functions would also be a solution. If we use real functions we form a linear combination of the sine and cosine function.

(1.15) $$x = A\cdot\cos\omega t + B\cdot\sin\omega t = C\sin(\omega t + \varphi) = C'\cos(\omega t + \varphi')$$

If we use complex solutions we bear in mind that ultimately we need a real solution for a real measurement. **We therefore use the real part or the imaginary part of a complex solution to describe a physical situation, which brings us back to** (1.15)**.**

When we impose the initial conditions, the two arbitrary constants A and B disappear.

We try as general solutions one of the following:

(1.16)
$$a) x(t) = A\cos\omega t + B\sin\omega t$$
$$b) x(t) = C\sin(\omega t + \varphi)$$
$$c) \hat{z}(t) = Ae^{i\omega t} + Be^{-i\omega t}$$
we use either the real part of the solution or the imaginary part. The initial conditions specify the location x(t=0) and the speed at v(t=0) at the time t=0.

When we start with a fully expanded spring, its initial location is evidently x(0)=A and its speed is v(0) = 0. These standard initial conditions lead immediately to the unique solution for the motion of the spring:

(1.17)
$$x(t) = A\cos\omega t \text{ with } \omega^2 = \frac{k}{m}$$
$$\mathrm{v}(t) = \dot{x} = -A\omega\sin\omega t;$$
$$a(t) = \ddot{x} = -\omega^2 x(t)$$

We can find the same solution when we consider the total energy of the spring:

(1.18)
$$E = \frac{1}{2}m\mathrm{v}^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

We know that the total energy of a simple spring with the force –kx is conserved.
We can use this information to directly find the solution for x and v:

(1.19)
$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$
For simplicity we assume the standard initial conditions $x(0)=x_0$ and v(0)=0
$$E=E(0)=\frac{1}{2}kx_0^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$
$$m\dot{x}^2 = kx_0^2 - kx^2$$
$$\dot{x}^2 = \frac{k}{m}\left(x_0^2 - x^2\right)$$

(1.20)
$$\left(\frac{dx}{dt}\right)^2 = \frac{k}{m}\left(x_0^2 - x^2\right) = \omega^2\left(x_0^2 - x^2\right)$$
$$\left(\frac{dt}{dx}\right)^2 = \frac{1}{\omega^2}\frac{1}{x_0^2 - x^2}; \frac{dt}{dx} = \pm\frac{1}{\omega}\frac{1}{\sqrt{x_0^2 - x^2}}$$
$$dt = \pm\frac{1}{\omega}\frac{dx}{\sqrt{x_0^2 - x^2}}$$
we choose the + sign, because we know that our answer must be fitted to the initial conditions.

(1.21)
$$t=\frac{1}{\omega}\sin^{-1}\left(\frac{x}{x_0}\right)+t_0 \Rightarrow \sin\left[\omega\left(t-t_0\right)\right]=\frac{x}{x_0}$$
$x=x_0\sin\left(\omega t-\omega t_0\right)$ According to the initial condition v(0)=0
$$\text{v}(0)=\omega x_0\cos\left(\underbrace{\cancel{\omega t}}_{t=0}-\omega t_0\right)=0\Rightarrow -\omega t_0=\frac{\pi}{2};$$
but $\sin(x+\frac{\pi}{2})=\cos x$

Therefore our solution is

(1.22)
$$x(t)=x_0\cos\omega t$$

It becomes a bit more challenging when we also have a **damping term b**:

(1.23)
$$a)F_{damping}=-b\text{v}(t)\equiv -b\dot{x}$$
The sum of the exterior forces on the spring is then:
$$b)F=-kx-b\dot{x}=m\ddot{x} \text{ or}$$
$$c)\ddot{x}+\frac{b}{m}\dot{x}+\frac{k}{m}x=0=\ddot{x}+\frac{b}{m}\dot{x}+\omega_0^2x$$

Neither a cosine nor a sine function (alone) will satisfy the d.e. in this case; a complex trial solution however leads to an algebraic condition, which is called the characteristic equation of the differential equation:

$a)\hat{z}(t)=\hat{z}_0e^{i\omega t}$ Remember that derivation means multiplication by i$\omega\Rightarrow$

(1.24) $b)\left(-\omega^2+i\omega\frac{b}{m}+\omega_0^2\right)\hat{z}(t)=0\Rightarrow -\omega^2+i\omega\frac{b}{m}+\omega_0^2=0$ quadratic equation in ω

$$\omega=\frac{-i\frac{b}{m}\pm\sqrt{-\frac{b^2}{m^2}+4\omega_0^2}}{-2}=i\frac{b}{2m}\pm\frac{1}{2}\sqrt{4\omega_0^2-\left(\frac{b}{m}\right)^2}$$

We distinguish three different possibilities, depending on the value of the radicand, the first of which is our most important case:

(1.25)
$$\omega=\text{i}\frac{b}{2m}\pm\sqrt{\omega_0{}^2-\left(\frac{b}{2m}\right)^2};\text{ for }\omega_0{}^2-\left(\frac{b}{2m}\right)^2>0$$

(1.26)
$$\omega=\text{i}\left(\frac{b}{2m}\pm\sqrt{\omega_0{}^2-\left(\frac{b}{2m}\right)^2}\right);\ \omega_0{}^2-\left(\frac{b}{2m}\right)^2<0$$

1 single solution:

(1.27)
$$\omega \text{ (double root)} = \mathrm{i}\frac{b}{2m};\ \omega_0^2 - \left(\frac{b}{2m}\right)^2 = 0$$

The original characteristic equation can be written as

$$\left(\omega - \mathrm{i}\frac{b}{2m}\right)^2 = 0 = \omega^2 - i\frac{b}{m}\omega - \left(\frac{b}{2m}\right)^2 = 0$$

The most important (for our purposes here) complex general solution to the differential equation corresponds to the case in (1.25):

(1.28)

$$a)\hat{z} = \hat{z}_0 e^{i\left(\mathrm{i}\frac{b}{2m} \pm \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}\right)t} = \hat{z}_0 e^{-\frac{b}{2m}t \pm i\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}\,t}$$ we can easily guess at a real solution x now (assume that the radicand is positive; remember the Euler formula):

$$\text{b)}x(t) = \mathrm{A}e^{-\frac{b}{2m}t}\cos\left(t\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}\right) = \mathrm{A}e^{-\frac{b}{2m}t}\cos\omega_1 t \text{ with } \omega_1 = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

We follow the general approach of trying to solve our equations with complex functions, and when we have found a general complex solution, we take the real part of that solution as our physical solution.(1.28) b) is an exponentially decreasing cosine function. We can rewrite the new frequency ω_1 in terms of the original ω_0^2=k/m, which we now call ω_0

(1.29)
$$a)\omega_1 = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} = \omega_0\sqrt{1 - \left(\frac{b}{2m\omega_0}\right)^2} = \omega_0\sqrt{1 - \frac{b^2}{4m^2}\frac{m}{k}} = \omega_0\sqrt{1 - \frac{mb^2}{m^2k4}} =$$
$$b)\omega_1 = \omega_0\sqrt{1 - \frac{b^2}{4km}};\ 1 - \frac{b^2}{4km} > 0$$

In first approximation $\omega_1=\omega_0$. This means that in damped oscillations the same frequency applies as long as the number $\frac{b^2}{4km}$ is small.

To prove this, it is useful to recall the binomial expansion formula (we used this already with the relativistic kinetic energy):

(1.30)
$$(1+x)^r = \sum_{k=0}^{\infty}\binom{r}{k}x^k \text{ with } \binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}$$
for k⟩0 and 0 for k=0

(1.31)
$$\boxed{(1+x)^r \approx 1 + r\cdot x \text{ for } |x| \ll 1}$$

thus (1.32) $(1-x)^{\frac{1}{2}} = 1-\frac{1}{2}x+\frac{1}{8}x^2; \binom{0.5}{1} = \frac{0.5}{1} = 0.5; \binom{0.5}{2} = \frac{0.5(0.5-1)}{2} = -\frac{1}{8}$

(1.33) $\omega_1 = \omega_0\sqrt{1-\frac{b^2}{4km}} \approx \omega_0\left(1-\frac{b^2}{8km}+\frac{1}{8}\left(\frac{b^2}{4km}\right)^2+++\right)$

Therefore, in first approximation, $\omega_1=\omega_0$.

Assume that k=2N/m; m=0.5kg; x(0)=1.2m. Find a damping factor b, which would cause the amplitude to decrease by 50% in 20 periods.

$$x(t) = Ae^{-\frac{b}{2m}t}\cos\omega_1 t;\ A = 1.2m;\ ;\omega_1 = \omega_0\sqrt{1-\frac{b^2}{4km}};\omega_0 = \sqrt{\frac{k}{m}} = 2s^{-1}$$

(1.34) We are looking for $0.6=1.2e^{-\frac{b}{2m}t}; t = 20T = 20\frac{2\pi}{\omega_0} = 20\pi$

$$b = -\frac{\ln 0.5}{20\pi} = 1.1\times10^{-2}\frac{Ns}{m};\omega_1 = \omega_0\sqrt{1-\frac{b^2}{4km}} \approx 2.00000s^{-1}$$

(1.35) $x(t)=Ae^{-\frac{b}{2m}t}\cos\left(t\sqrt{\omega_0{}^2-\left(\frac{b}{2m}\right)^2}\right) = Ae^{-\frac{b}{2m}t}\cos\omega_1 t$ with $\omega_1 = \sqrt{\omega_0{}^2-\left(\frac{b}{2m}\right)^2}$

Find the time it takes for the amplitude to decrease by the factor 2:

(1.36)
$$0.5A = Ae^{-\frac{b}{2m}t};\ \frac{1}{2}=e^{-\frac{b}{2m}t};\ -\ln2=-\frac{b}{2m}t$$
$$b = \frac{2m\ln 2}{t_{1/2}}$$

Thus, by measuring the time it takes for a spring oscillation to have its amplitude reduced to half the original value, we can experimentally determine the damping factor b.

Now, what happens to the energy? As we have friction, we do not expect the total energy to remain constant. It loses energy over time. Let us find out how:

$$\frac{dE}{dt} = mv\frac{dv}{dt}+kx\frac{dx}{dt}$$

(1.37) $m\frac{dv}{dt} = -kx-b\frac{dx}{dt}$ is the d.e. which we substitute in the formula for dE

$$\frac{dE}{dt} = \frac{dx}{dt}\left(-kx-b\dot{x}\right)+kx\frac{dx}{dt} = -bv^2$$

This means that a damped oscillation does not conserve energy but loses it at the rate of bv^2.

Remarks on the special case (optional) in which the $\omega_1 = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$ =0. In this case our approach yields only a single solution to the d.e., namely $x_0 e^{-\frac{b}{2m}t}$. However, according to the theory of differential equations a linear differential equation of the second order has two separate solutions, which combine to a single solution if one applies the initial conditions. This second solution is in the present case given by $x_1 t e^{-\frac{b}{2m}t}$. The general solution in this case is therefore:

(1.38) $$x(t) = (x_0 + x_1 t)e^{-\frac{b}{2m}t}$$ in which x_0 and x_1 are constants determined by the initial conditions.

In the case of the standard initial conditions x(0)=A, v(0)=0 this yields

(1.39) $$x_0 = A \text{ and } x_1 = \frac{b}{2m}A$$
$$x(t) = \left(1 + \frac{b}{2m}t\right)Ae^{-\frac{b}{2m}t}$$

One can check this by inserting:

(1.40) $$x(t) = \left(1 + \frac{b}{2m}t\right)Ae^{-\frac{b}{2m}t} = A \text{ for t=0}$$
$$\dot{x} = -\frac{b}{2m}Ae^{-\frac{b}{2m}t} + \frac{b}{2m}Ae^{-\frac{b}{2m}t} - \left(\frac{b}{2m}\right)^2 tAe^{-\frac{b}{2m}t} = -\left(\frac{b}{2m}\right)^2 tAe^{-\frac{b}{2m}t} = 0 \text{ for t=0}$$

Forced Oscillations:

It very often happens that an oscillation is driven from the outside, i.e. that there is an additional force applied to the spring, for example, an electric oscillation with a different frequency, which we now label ω_f.

This outside force may have the form $F\cdot\cos\omega_f t$, with ω_f being the ***driving*** frequency. We refer to the ***ω_0 as the intrinsic or natural frequency***, the frequency determined by k/m in the case of the spring. One can easily see that if this outside force is applied long enough, the spring will oscillate more or less with the new forced frequency ω_f.

Mathematically we have to solve the following situation:

(1.41)

$a)\ddot{x}+\frac{k}{m}x=\frac{F}{m}\cos\omega_f t$ We are only looking for a special solution after a long time, when the forced system oscillates with the frequency ω_f. We try a new exponential solution and write the right hand side of equation a) in terms of a complex function: $\cos\omega_f t \Rightarrow e^{i\omega_f t}$

b) $\hat{x}=Ae^{i\omega_f t}$ we get $\left(-\omega_f{}^2+\omega_0{}^2\right)Ae^{i\omega_f t}=\frac{F}{m}e^{i\omega_f t}$ which means that $A=\frac{F}{m\left(-\omega_f{}^2+\omega_0{}^2\right)}$

This means that we get a new oscillation with the amplitude:

$c)A\left(\omega_f\right)=\frac{F}{m\left(\omega_0{}^2-\omega_f{}^2\right)}$; this means that the amplitude is a function of the frequency of the applied exterior force.

The new amplitude grows to infinity as the forced frequency ω_f approaches the natural frequency ω_0. This is what we call a resonance effect. In reality, there is always a damping factor b/m, which appears in the parenthesis of the denominator in (1.41) so that $A(\omega_f)$ does not go to infinity but can still grow to extremely large (and often destructive) values.

If we use the damped equation, we proceed as follows to find the special solution for very large values of t:

(1.42) $$Set\ \hat{x}=Ae^{i\omega_f t}$$

$$\ddot{x}+\frac{b}{m}\dot{x}+\frac{k}{m}x=\frac{F}{m}e^{i\omega_f t}\Rightarrow\left(-\omega_f{}^2+i\omega_f\frac{b}{m}+\frac{k}{m}\right)Ae^{i\omega_f t}=\frac{F}{m}e^{i\omega_f t}\Rightarrow$$

(1.43) $$\hat{A}=\frac{F/m}{\left(\omega_0^2-\omega_f{}^2+i\omega_f\frac{b}{m}\right)}=re^{-i\beta}=\frac{F/m}{\sqrt{\left(\omega_0^2-\omega_f{}^2\right)^2+\left(\omega_f\frac{b}{m}\right)^2}}e^{-i\beta};\ \beta=\tan^{-1}\frac{b_1}{a_1}$$

where a_1 is the real part and b_1 is the imaginary part in the complex function:

$$\hat{z}=\underbrace{\omega_0^2-\omega_f{}^2}_{a_1}+i\underbrace{\omega_f\frac{b}{m}}_{b_1}=a_1+ib_1$$

Practice example to rationalize the complex denominator of a fraction:

(1.44) $$\frac{1}{3+4i}=\frac{1}{\sqrt{3^2+4^2}e^{i\left(Arc\tan\frac{4}{3}+2n\pi\right)}}=\frac{1}{5e^{i\left(Arc\tan\frac{4}{3}+2n\pi\right)}}=\frac{1}{5e^{i(0.9274+2n\pi)}}=\frac{1}{5}e^{-i(0.9274+2n\pi)}$$

We rationalize the complex amplitude according to (1.7)

(1.45) $\hat{A}=\frac{1}{a_1+ib_1}=\frac{1}{\sqrt{a_1^2+b_1^2}}e^{-i\beta}$ different meaning for b_1; dont confuse with b, the air drag coefficient!!

(1.46)
$$\hat{A}(\omega_f)=\frac{F/m}{\sqrt{(\omega_0^2-\omega_f^2)^2+\frac{\omega_f^2b^2}{m^2}}}e^{-i\beta}$$
$$\hat{x}=\hat{A}(\omega_f)e^{i(\omega_f t-\beta)}=\frac{F/m}{\sqrt{(\omega_0^2-\omega_f^2)^2+\frac{\omega_f^2b^2}{m^2}}}e^{i(\omega_f t-\beta)}$$ particular solution for large t

This means that the amplitude of the forced oscillation with damping is a function of the frequency ω_f of the driving force. If this frequency approaches the original frequency of the spring, the amplitude will approach its maximum value. We call this effect resonance. The original frequency is referred to as resonance frequency.

As shown earlier a damped oscillation loses energy at the rate of

(1.47) $$\frac{dE}{dt}=-b\mathrm{v}^2$$

One can prove this easily by considering the total energy and then using the differential equation for damped oscillation (see earlier (1.37)):

(1.48)
$$E=\frac{1}{2}kx^2+\frac{1}{2}m\mathrm{v}^2;\ m\ddot{x}+b\dot{x}+kx=0;\ m\ddot{x}=-(b\dot{x}+kx)$$
$$\frac{dE}{dt}=kx\dot{x}+m\mathrm{v}\dot{\mathrm{v}}=kx\mathrm{v}+\mathrm{v}m\ddot{x}=kx\dot{x}+\mathrm{v}(-b\dot{x}-kx)=-b\mathrm{v}^2$$

This is evidently much easier than finding the energy from the solution $x=x_0\mathrm{e}^{-\frac{b}{2m}t}\cos\omega_1 t$:

Let us use the example below to calculate some numbers for the energy loss:

(1.49)
$$x=x_0\mathrm{e}^{-\frac{b}{2m}t}\cos\omega_1 t\approx x_0\cos\omega_0 t$$
$$\omega_1=\sqrt{\omega_0^2-\left(\frac{b}{2m}\right)^2}$$

(1.50) $$\mathrm{v}_{\max}=x_0\omega_0$$

Use 10 cm for the amplitude of your oscillation and a mass of 1kg. Calculate ω_0 and choose values for $\omega_0 t$ between 0 and 2π.

$$\omega_1 \approx \omega_0 = \sqrt{\frac{k}{m}} = 5.2 s^{-1}$$

If b=1.00E-4 we get for the maximum energy loss per second

$$-b\mathrm{v}^2 \approx -b\left(A\omega_0\right)^2 = -1.0\times10^{-4}\times0.27 = -2.7\times10^{-5}\,Watts$$

In quantum physics we find that the energy emitted by any oscillator is a multiple of the angular frequency and Planck's quantum ħ

The energy for N oscillators is therefore $N\hbar\omega_0$ with $\hbar \approx 10^{-34}\,Js$

(Reminder: We have seen elsewhere that the angular momentum of the electronic orbits in atoms is also quantized: $l = n\hbar$)

This means that per second the oscillator above emits about $4\text{x}10^{28}$ energy quanta. We also find the result in quantum physics that the minimal energy of an oscillator is not 0 but $\frac{1}{2}\hbar\omega_0$. This is consistent with Heisenberg's uncertainty relation which does not allow any object to be totally at rest. It also illustrates the fact that Heisenberg's uncertainty relation is not merely an expression about the uncertainty of measurement but about the indeterminate nature of actuality itself.

Additional fun stuff (optional)

Wavenumber k and derivatives (optional):

We have seen that derivatives are particularly easy to obtain with complex exponential functions. It does not matter whether we talk about time derivatives, spatial derivatives or partial derivates:

$f(x) = A\sin kx$; k is the wavenumber, the number of complete wave-cycles in the spatial x-directionfitting within 2π.

(1.51) The wavenumber is for the space coordinates what ω is for the time coordinate. (A sine or cosine wave function with amplitude "A" can be described as the projection of circular motion with radius "A" on the x or y axis, ω is also the angular frequency.)

$$\frac{df}{dx} = kA\cos kx$$

$$k = \frac{2\pi}{\lambda}; \omega = \frac{2\pi}{T}$$

The wave number k can become a vector, when we consider a sine function in three dimensions:

(1.52)
$$\sin(k_x x + k_y y + k_z z)$$
$$k_x = \frac{2\pi}{\lambda_x}; k_y = \frac{2\pi}{\lambda_y}; k_z = \frac{2\pi}{\lambda_z}; \vec{k} = \langle k_x, k_y, k_z \rangle$$
$$\vec{k} \cdot \vec{r} = k_x x + k_y y + k_z z$$

All functions $f(kx - \omega t + \Phi)$ represent linear waves. We discuss this in the next chapters 16-18. This includes complex functions which can be simply thought of as being combinations of sine and cosine functions.

$$Let\ \hat{f}(x,t) = Ae^{i(kx-\omega t)}$$
$$\frac{\partial \hat{f}}{\partial t} = -i\omega \hat{f}(x,t) \text{ and } \frac{\partial \hat{f}}{\partial x} = ik\hat{f}(x,t)$$

(1.53)
$$Let\ \hat{f}(x,y,z,t) = Ae^{i\left(\vec{k}\vec{r} - \omega t\right)} \Rightarrow \vec{\nabla}\hat{f}(\vec{r},t) = grad \cdot \hat{f}(\vec{r},t) = i\vec{k}\hat{f}(\vec{r},t)$$

We see that for exponential function the del operator $\vec{\nabla}$ becomes $i\vec{k}$