Homework: See website.

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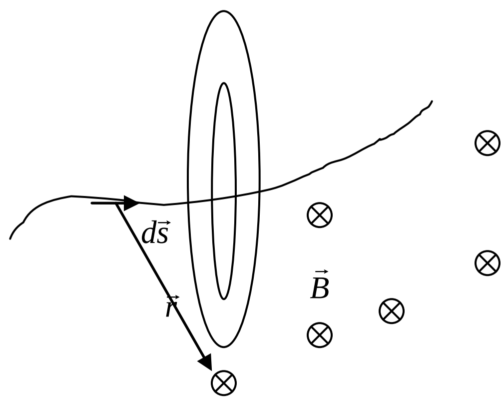
30.1 The Biot-Savart Law, or how to calculate the magnetic field of a current.

Biot and Savart (19th century) determined a mathematical expression for calculating the magnetic field of a current:

The known experimental facts were:



a)Around any current carrying wire curls a magnetic field. The field vectors point into the plane below the wire and come out of the plane above the wire. (Any moving charge has a magnetic field around it.)

b) The magnitude of $d\vec{B}$ is proportional to $1/r^2$, where r is the distance from the line element $d\vec{s}$ to the point where we calculate the magnetic field.

c) The magnitude of $d\vec{B}$ is proportional to the current and to the length ds.

d) The magnitude dB is proportional also to sinθ, where θ is the angle between the vectors $d\vec{s}$ *and* $\vec{r}$

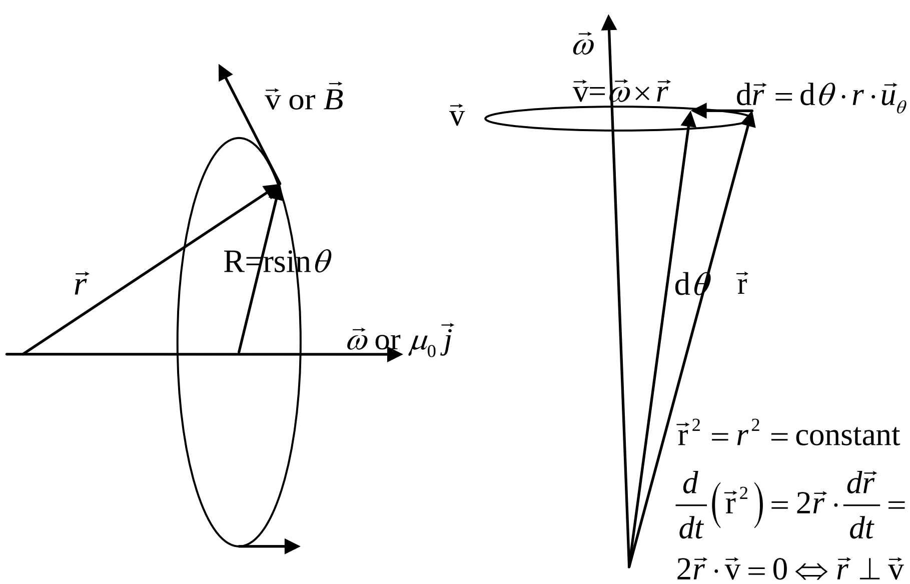
Using these experimental data and our knowledge of other relationships from mechanics in which one vector curls around another, let us try to guess at the mathematical relationship between currents and magnetic fields.

This information can be summarized by the statement that the magnetic field $\vec{B}$ curls around the current density $\vec{j}$. In physics this can generally be expressed by the differential statement:

(30.0) $$curl\vec{B} \text{ is proportional to the current density } \vec{j}, or, \vec{\nabla}\times\vec{B} \propto \vec{j}$$

30.1a The curl of the velocity vector field**:**

In mechanics we had a typical situation in the velocity of circular motion. The velocity vector $\vec{v}$ curls around the vector for angular velocity $\vec{\omega}$. We described the relationship by:



(30.1)

$$\vec{v}=\vec{\omega}\times\vec{r}\Rightarrow v=\omega r\sin\theta$$
$$\vec{v}\perp\vec{\omega} \text{ and } \vec{v}\perp\vec{r}$$

This relationship can also be captured in an interesting differential form, namely :

(30.2) $$\vec{\nabla} \times \vec{v} \equiv \text{curl } \vec{v} = 2\vec{\omega}$$

By proving that this is always correct for any two vectors curling around each other in the way described we can have a guess at the relationship between the current and the magnetic field, namely that the curl of the magnetic field B is proportional to the current density:
This relationship is called Ampere's law.

(30.3) $$\boxed{\overrightarrow{curl\ B} = \vec{\nabla} \times \vec{B} = \mu_0 \vec{j}}$$

Let us first prove that (30.2) is correct.
Using the properties of the cross product we get:

(30.4) $$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = \vec{i}\left(z\omega_y - y\omega_z\right) - \vec{j}\left(z\omega_x - x\omega_z\right) + \vec{k}\left(y\omega_x - x\omega_y\right)$$

Now we are ready to calculate the curl of the velocity vector in circular motions:

(30.5) $$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ z\omega_y - y\omega_z & x\omega_z - z\omega_x & y\omega_x - x\omega_y \end{vmatrix} = \vec{i}\left(\omega_x + \omega_x\right) + \vec{j}\,2\omega_y + \vec{k}\,2\omega_z = 2\vec{\omega}$$

Thus, the equation (30.3) must be correct. The law is called Ampere's law, the coefficient is μ_0 and is called the **permeability of free space**:

(30.6) $$\boxed{\mu_0 = 4\pi \cdot 10^{-7} \frac{Tm}{A};\ \frac{\mu_0}{4\pi} = 10^{-7} \frac{Tm}{A}}$$

(Memorize the number!)
As we shall see later there is a close relationship between these constants appearing in the magnetic field and in the electric field.

(30.7) $$\boxed{\mu_0 \varepsilon_0 c^2 = 1;\ \varepsilon_0 = 8.854 \cdot 10^{-12};\ k_e = \frac{1}{4\pi\varepsilon_0} = 8.98 \cdot 10^{8};\ \mu_0 = 1.257 \cdot 10^{-6};\ \frac{\mu_0}{4\pi} = 10^{-7}}$$

Comparing the formulas (30.2) and (30.3) we suspect that the current density plays the role of ω in (30.1), thus we guess:

(30.8)
$$\vec{\nabla} \times \vec{v} = 2\vec{\omega} \ and\ \vec{v} = \vec{\omega} \times \vec{r} \ \Rightarrow \vec{\nabla} \times \vec{B} = \mu_0 \vec{j} \ and\ \vec{B} \propto \mu_0 \vec{j} \times \vec{r}$$
$$\vec{B} \text{ is proportional to } \vec{j} \times \vec{r}.\ with\ \vec{j} = n_q q \vec{v}$$

The correct answer as to the mathematical expression for an infinitesimal magnetic field created by an infinitesimal current segment is given by:

(30.9) $$d\vec{B} = k_B \frac{Id\vec{s} \times \vec{u}_r}{r^2}$$

In both electric and magnetic fields one finds experimentally a decrease of the magnitude of the various fields with $1/r^2$.
The remaining constant can be double checked through measurement, and we find the constant is equal to

(30.10) $$k_B = \frac{\mu_0}{4\pi}$$

In its final form this law is known as the law of Biot and Savart :

(30.11) $$d\vec{B}(r) = \frac{\mu_0 I}{4\pi}\frac{d\vec{s} \times \vec{r}}{r^3} = \frac{\mu_0 I}{4\pi}\frac{d\vec{s} \times \vec{u}_r}{r^2} = \frac{\mu_0 I_1}{4\pi}\frac{d\vec{s}_1 \times \vec{u}_r}{|\vec{r} - \vec{r}_1|^2}$$

We get the total field by integrating over the whole length of the current carrying wire.
The unit vector $\vec{u}_r$ points from the location $\vec{r}_1 = \langle x_1, y_1, z_1 \rangle$ of the moving charges $dQ \cdot \vec{v} = Id\vec{s}$ to the point $\vec{r} = \langle x, y, z \rangle$ where we calculate the magnetic field. $\vec{u}_r = \frac{\vec{r} - \vec{r}_1}{|\vec{r} - \vec{r}_1|}$

It is often useful to give the charges dQ and their velocity the index 1, as a reminder that the location of the current segment (the charge density, the charge etc) is different from the location where we calculate the magnetic field. This is what we also did in the calculation of the electric field and the electric potential. The charges create fields everywhere in space. Static charges create electric fields, moving charges create additional magnetic fields.

(There is a direct derivation of the law of Biot-Savart based on Ampere's law in the addendum part of the lecture notes of the previous chapter 29. Most text books don't give you this derivation and start simply with a postulate for the law of Biot-Savart.)

30.1b Law of Biot-Savart for a single charge:
For a single charge q moving with velocity $\vec{v}$ we just need to replace

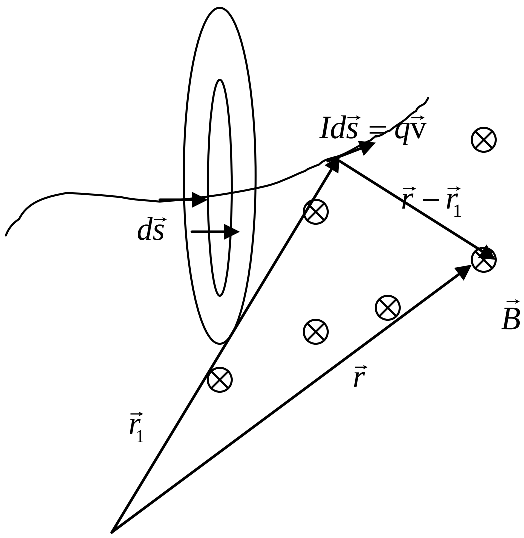
(30.12)
$$I \cdot d\vec{s} \text{ with } q\vec{v}$$
$$\frac{dQ}{dt} \cdot d\vec{s} = \mathrm{dQ} \cdot \frac{d\vec{s}}{dt} \Rightarrow q \cdot \frac{d\vec{s}}{dt} = q\vec{v}$$

and get the total magnetic field, not just the infinitesimal one, which requires integration over the whole length of current.

(30.13) $$\vec{B}(r) = \frac{\mu_0 q}{4\pi}\frac{\vec{v} \times \vec{r}}{r^3} = \frac{\mu_0 q}{4\pi}\frac{\vec{v} \times \vec{u}_r}{r^2} = \frac{\mu_0 q}{4\pi}\frac{\vec{v} \times \vec{u}}{|\vec{r} - \vec{r}_1|^2}$$

If the charge q is located at the origin, the vector $\vec{r}_1$ is obviously $\vec{0}$.

We use the same convention as when we calculated the electric field. The location where we calculate the magnetic field is given by $\vec{r}$. The location of the moving charge $q\vec{v}$ *or* $(Id\vec{s})$ is given by $\vec{r}_1$. The vector from the moving charge to the location of the magnetic field is given by $\vec{r}-\vec{r}_1$.The unit vector $\vec{B}$



$$\vec{u}_r=\frac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|}=\frac{\langle x-x_1,y-y_1\rangle_1}{\sqrt{(x-x_1)^2+(y-y_1)^2}}$$

points from the moving charge q to the location of the magnetic field $\vec{B}(r)$.

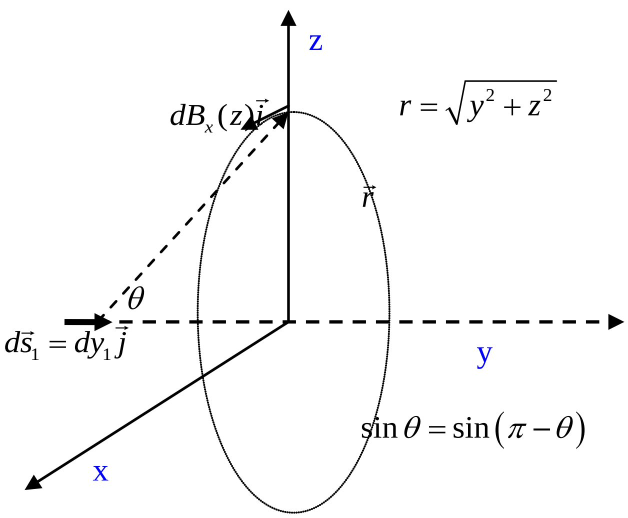
Example: Magnetic Field of the electron in the Bohr Atom:
An electron moving in a circular orbit around the proton (from Bohr's model of the atom) creates a magnetic field of 12.5T at the center of the atom, which is a huge magnetic field. (See problem 1 of this chapter.) In that case of circular motion the relationship between Ids and qv becomes nice and clear: The time interval dt becomes the period T; ds=rdθ=r2π

(30.13) $$qv=q\frac{2\pi r}{T}=Ids=\frac{dQ}{dt}ds=\frac{q}{T}2\pi r$$

The magnetic field curls around the orbit of the electron and creates a resultant magnetic field at the location of the proton. It is perpendicular to the orbital plane of the electron.

(30.14) $$B=\frac{\mu_0 q}{4\pi}\frac{\vec{v}\times\vec{r}}{r^3}=\frac{\mu_0 q v}{4\pi r^2}=12.5T;v=2.19\cdot 10^6\frac{m}{s};r=0.529\cdot 10^{-10}m$$

<u>30.1c Calculation of the magnetic field for various situations</u>:
Example 1: Let us calculate the magnetic field created by a long straight wire. Obviously, the magnetic field will circle around it. If we assign the direction of the current to the positive y-direction going to the right, the magnetic fieldlines will go in the positive x direction B_x above the wire and the negative x direction below the wire. $d\vec{s}$ and $\vec{r}$ lie in the y-z plane, consequently, $d\vec{B}$ is perpendicular to that plane, in the x direction. The vector $\vec{r}$ points from the location of the current segment to the location where we calculate the magnetic field. (To do this more consistently we should draw the vector as $\vec{r}-\vec{r}_1$, $\vec{r}$ being the vector from 0 to the location of B, and $\vec{r}_1$ being the vector from 0 to the location of the charge q_1 or the current segment $d\vec{s}_1$.) Using



the right hand rule we draw the appropriate picture of the coordinates.
(30.15)

$$dB_x\vec{i} = \frac{\mu_0}{4\pi} I \frac{dy_1 \cdot \vec{j}\times\vec{u}_r}{r^2} = \frac{\mu_0}{4\pi}\frac{Idy_1\sin\theta}{y^2+z^2}\vec{i}$$

(Note that $\vec{i}$ and $\vec{j}$ are unit vectors, not currents.)
The magnitude of the cross-product in the numerator $d\vec{s}\times\vec{u}_r$ is equal to the product of the perpendicular components of these quantities, namely $dy_1\sin\theta$:

also: $z = r\sin\theta$ or $r = \dfrac{z}{\sin\theta}$

We need a relationship between dy_1 and $d\theta$:

$$y_1 = z\cot\theta; dy_1 = -z\cdot\operatorname{cosec}^2\theta\cdot d\theta = -\frac{1}{\sin^2\theta} zd\theta \quad -\infty < y_1 < +\infty \Rightarrow \pi \le \cot\theta \le 0$$

(Reminder: $\dfrac{d}{d\theta}\cot\theta = -\operatorname{cosec}^2\theta\cdot d\theta = \dfrac{-d\theta}{\sin^2\theta}$; $\dfrac{d}{d\theta}\tan\theta = \sec^2\theta\cdot d\theta = \dfrac{d\theta}{\cos^2\theta}$)

The integrand can therefore be written as:

(30.16)
$$\frac{dy_1\sin\theta}{r^2} = \underbrace{\frac{-zd\theta}{\sin^2\theta}}_{dy}\sin\theta\underbrace{\frac{1}{z^2/\sin^2\theta}}_{r^2} = \frac{-z\sin\theta d\theta}{z^2} = \frac{-\sin\theta d\theta}{z}$$

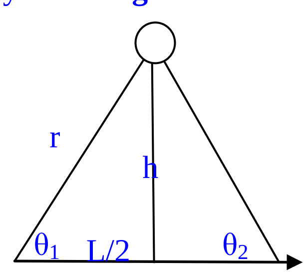
We have the total integration from $\theta=\pi$ with the current element to the far left ($y_1=-\infty$)to $\theta=0$ for the current element to the far right at $y=+\infty$.

(30.17)
$$B_x = \frac{\mu_0 I}{4\pi}\int_\pi^0 \frac{-\sin\theta d\theta}{z} = \frac{\mu_0 I}{4\pi}\frac{1}{z}(\cos 0 - \cos\pi) = \frac{\mu_0}{2\pi}\frac{I}{z}$$

We often prefer to call the perpendicular distance from the wire r, and therefore write this important result as: The magnetic field created by a long wire with current I circles around the wire and has the magnitude:

(30.16)
$$\boxed{B = \frac{\mu_0}{4\pi}\frac{2I}{r} = \frac{\mu_0}{2\pi}\frac{I}{r}}$$

Any **short segment of a straight wire of length L, carrying a current I in the counter clockwise direction, creates a magnetic field on the bisector at height h**. This magnetic field is perpendicular to the page and its magnitude is given by:

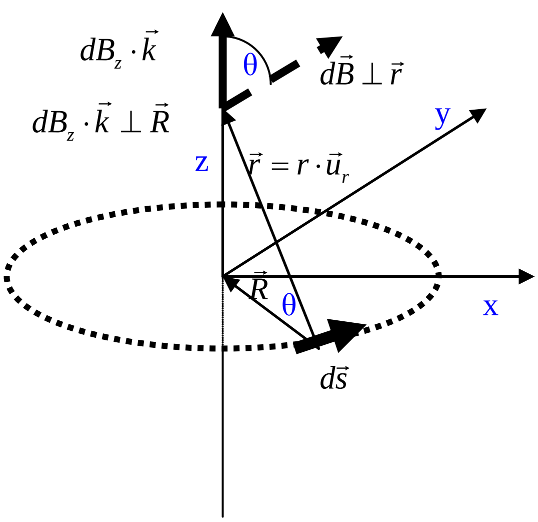


(30.17)
$$B = \frac{\mu_0 I}{4\pi h}\left(\cos\theta_1 - \cos\theta_2\right) = \frac{\mu_0 I}{4\pi h}\left(\cos\theta_1 - \underbrace{\cos\left(\pi - \theta_1\right)}_{-\cos\theta_1}\right)$$

$$B = \frac{\mu_0 I}{4\pi h} 2\cos\theta_1; \cos\theta_1 = \frac{L/2}{r}$$

(30.18)
$$\boxed{B = \frac{\mu_0 I}{4\pi h} 2\cos\theta_1}$$

We could also have used the following approach:



(30.19)
$$dB_x \vec{i} = \frac{\mu_0}{4\pi} I \frac{dy \cdot \vec{j} \times \vec{u}_r}{r^2} = \frac{\mu_0}{4\pi}\frac{Idy\sin\theta}{y^2+z^2}\vec{i} =$$
$$\frac{\mu_0}{4\pi}\frac{Idy}{y^2+z^2}\frac{z}{\sqrt{y^2+z^2}}\vec{i} = \frac{\mu_0}{4\pi}\frac{zIdy}{\left(y^2+z^2\right)^{\frac{3}{2}}}\vec{i}$$

The integral needed is given by:
(30.20)
$$z\int_{-\infty}^{\infty}\frac{dy}{\left(z^2+y^2\right)^{\frac{3}{2}}} = z\frac{y}{z^2\sqrt{z^2+y^2}}\Bigg|_{-\infty}^{\infty} = \frac{1}{z}[1+1];$$

$$\lim_{y\to\infty}\frac{y}{\sqrt{z^2+y^2}} = \frac{y}{\sqrt{\cancel{z^2}+y^2}} = 1$$

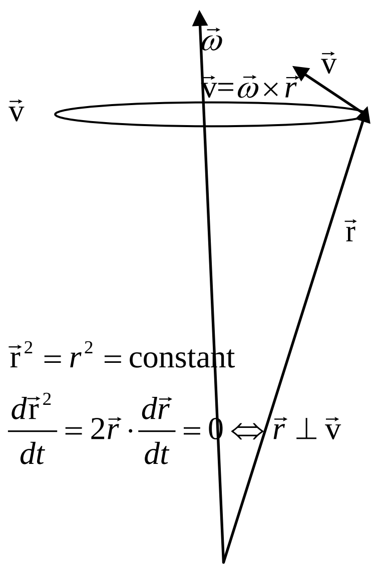
Example 2: Magnetic Field Created by a Magnetic Dipole.
In the previous chapter we saw that a current loop inserted in a magnetic field experiences a torque. We defined the magnetic moment of N loops:
(30.21)
$$\vec{\mu} = NI\vec{A}$$
The torque experienced by such a loop in a magnetic field equals :
(30.22)
$$\vec{\tau} = \vec{\mu} \times \vec{B}$$
and its potential energy is :
(30.23) $U = -\vec{\mu}\vec{B}$
A magnetic moment, i.e. a loop of current creates of course also a magnetic field.

When we calculate the magnetic field created by a current along a circular loop on the central axis, we need to know the angle between the velocity of the charges in the loop and the radius vector pointing to the central axis along which we need to calculate the magnetic field. The angular relationships are the same as in the relationship between velocity, angular velocity, and radius vector.

(30.24) $\vec{v}=\vec{\omega}\times\vec{r}$ like in $d\vec{B}=\frac{\mu_0}{4\pi}\frac{I}{r^3}d\vec{s}\times\vec{r}$

Find the angle between $d\vec{s}$ and $\vec{r}$:

In any such relationship the vector $d\vec{s}$ lies in the plane defined by the circular motion and the vector $\vec{r}$ points from the velocity vector to the location of $\vec{B}$ on the central axis.



$\vec{r}^2 = r^2 = \text{constant}$

$\frac{d\vec{r}^2}{dt} = 2\vec{r}\cdot\frac{d\vec{r}}{dt} = 0 \Leftrightarrow \vec{r}\perp\vec{v}$

Now, $d\vec{s}\perp\vec{r}$ just like $\vec{v}\perp\vec{r}$, Proof: $\vec{r}\cdot\vec{v}=\vec{r}\cdot(\vec{\omega}\times\vec{r})=0$ (if two vectors in a mixed product are the same, or are parallel , the mixed product is 0.)

The vector $d\vec{B}\perp\vec{r}$ and $\perp$ to the plane defined by $d\vec{s}$ *and* $\vec{r}$.

(Reminder of the properties of the cross product:

$$\vec{C}=\vec{A}\times\vec{B}\Rightarrow\vec{C}\cdot\vec{A}=\left(\vec{A}\times\vec{B}\right)\cdot\vec{A}=0$$

$$\Rightarrow\vec{C}\cdot\vec{B}=\left(\vec{A}\times\vec{B}\right)\cdot\vec{B}=0$$

Let us calculate the B-field on the axis perpendicular through this current carrying loop.

By symmetry we see that only the z components of the field will add up. All other components cancel as we go around the loop. So we calculate the magnitude of the B field and then project it on the z-axis. As $d\vec{s}$ is tangential to the circle the angle between it and $\vec{r}$ is a right angle. Therefore the magnitude of the cross product is:

$\left|d\vec{s}\times\vec{u}_r\right| = ds\cdot 1\cdot\sin\theta = ds$

(30.25) $$d\vec{B}=\frac{\mu_0 I}{4\pi}\frac{d\vec{s}\times\vec{u}_r}{r^2};dB=\frac{\mu_0 I}{4\pi}\frac{ds}{r^2}$$

(30.19) $$r^2 = R^2 + z^2$$ $$dB_z = dB\cos\theta$$

Look at the angle θ between the z-axis and $d\vec{B}$.

As the vector $d\vec{B}$ is perpendicular to the vector $\vec{r}$, the angle θ is also the angle between $\vec{r}$ and $\vec{R}$. (Angles with their arms perpendicular to each other are the same.)

$\cos\theta = \frac{R}{r} = \frac{R}{\sqrt{R^2+z^2}}$

The integral becomes:

(30.27) $$B_z = \frac{\mu_0 I}{4\pi}\oint \frac{\cos\theta ds}{r^2} = \oint \frac{\mu_0 I}{4\pi}\frac{Rds}{\left(R^2+z^2\right)^{\frac{3}{2}}} = \frac{\mu_0 I}{4\pi}\frac{R}{\left(R^2+z^2\right)^{\frac{3}{2}}}\underbrace{\oint ds}_{2\pi R} = \frac{\mu_0 I}{4\pi}\frac{R}{\left(R^2+z^2\right)^{\frac{3}{2}}}2\pi R$$

We integrated ds around the circular loop of radius R. Note the differences between R and r.

(30.28) $$\boxed{B = \frac{\mu_0 I}{2}\frac{R^2}{\left(R^2+z^2\right)^{\frac{3}{2}}}}$$

This is the magnetic field B_z along the central z-axis.

At the **center of a loop** with N coils we get z=0 and consequently:

(30.29) $$\boxed{B = \frac{\mu_0 NI}{2R}}$$

Compare this to the direct calculation from (30.13) for an electron in the orbit around a proton. There, the current I simply becomes q/T:

(30.30) $$\vec{B}(r) = \frac{\mu_0 q}{4\pi}\frac{\vec{v}\times\vec{u}_r}{r^2} \Rightarrow B = \frac{\mu_0 q}{4\pi}\frac{2\pi r/T}{r^2} = \frac{\mu_0 q}{2RT}$$

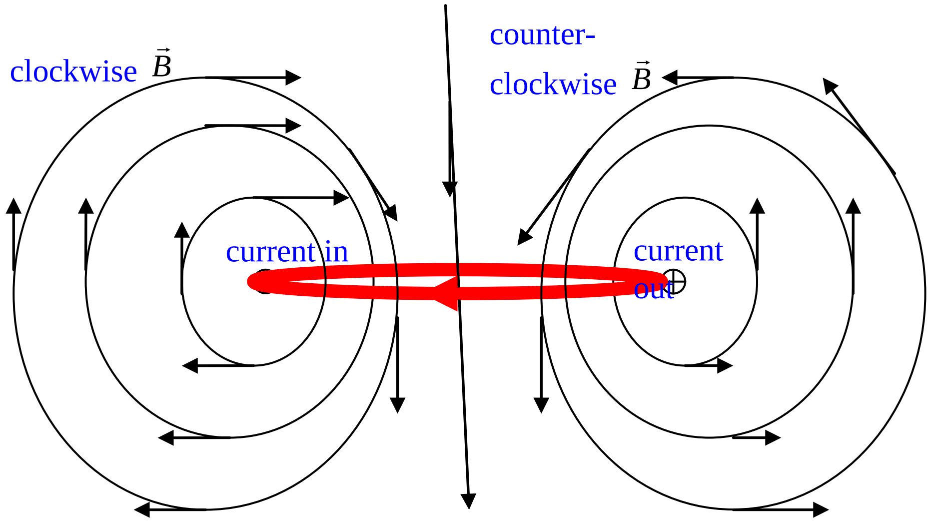
At a **great distance from the loop**, R becomes negligible in the denominator and we get for the magnetic field on the z-axis:

(30.31) $$\boxed{B = \frac{\mu_0 I}{2}\frac{R^2}{z^3}}$$

Note that the integration took place over the closed loop of radius R, in which the current is running. The magnetic field at the point P depends only on the fixed distances z and R. Note also that the magnetic field lines around a circular loop of currents resembles very much the field lines around a permanent bar magnet, which suggests, what has been stated before, namely that all magnetic fields without exception arise from currents. There are no magnetic monopoles.

$$curl\vec{B} = \mu_0 \vec{j} \text{ and } div\vec{B} = 0$$

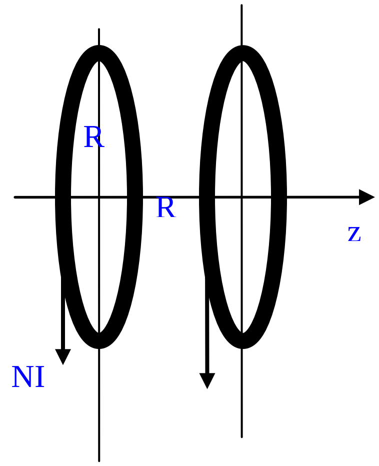
We only calculated the magnetic field line on the central axis through a loop of current, which forms a closed loop at infinity. We expect that, as we get away from the central axis, the magnetic fields loop around the current. For a few field lines it would approximately look like



this.

Helmholtz coils:

Now, calculate the magnetic field between two coils with N turns of equal size and with equal currents, an arrangement called Helmholtz coils. The distance between the two coil centers is equal to the radius R of either coil. We can show that in the center between these two coils the magnetic field is uniform.



If we put the origin of the z axis into the center of the left coil, the left coil creates a magnetic field at the location z according to formula . The field created by the second coil will be at location z'=z-R. (For z' =0 we are at z=R.) Adding up the fields of the two coils we get:

(30.31)

$$B_{res} = B_1 + B_2 = \frac{\mu_0 NI}{2} R^2 \left(\frac{1}{\left(R^2 + z^2\right)^{\frac{3}{2}}} + \frac{1}{\left(R^2 + (z-R)^2\right)^{\frac{3}{2}}} \right) =$$

$$\frac{\mu_0 NI}{2} R^2 \left(\frac{1}{\left(R^2 + z^2\right)^{\frac{3}{2}}} + \frac{1}{\left(2R^2 + z^2 - 2Rz\right)^{\frac{3}{2}}} \right)$$

To show that this resultant field is uniform at the midpoint z=R/2 between the two coil centers, we need to check whether its change with respect to z is 0, $\frac{dB_{res}}{dz} = 0$? The second derivative must also be equal to 0 in order to ensure that we don't have a maximum or minimum at that location.

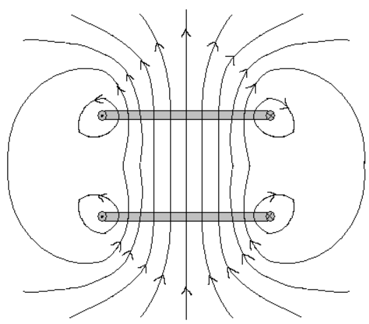
(30.31)

$$\frac{d}{dz}\left(\frac{1}{\left(R^2+z^2\right)^{\frac{3}{2}}}+\frac{1}{\left(2R^2+z^2-2Rz\right)^{\frac{3}{2}}}\right)=-\frac{3}{2}\left(\frac{2z}{\left(R^2+z^2\right)^{\frac{5}{2}}}+\frac{2z-2R}{\left(2R^2+z^2-2Rz\right)^{\frac{5}{2}}}\right)$$;for z= $\frac{R}{2}$ we get:

$$-\frac{3}{2}\left(\frac{R}{\left(R^2+\frac{R^2}{4}\right)^{\frac{5}{2}}}+\frac{R-2R}{\left(2R^2+\frac{R^2}{4}-R^2\right)^{\frac{5}{2}}}\right)=-\frac{3}{2}\left(\frac{R}{\left(R^2+\frac{R^2}{4}\right)^{\frac{5}{2}}}-\frac{R}{\left(R^2+\frac{R^2}{4}\right)^{\frac{5}{2}}}\right)=0$$

The same thing happens for the second derivative, i.e. both $\frac{dB_{res}}{dz}=0$ and $\frac{dB^2{}_{res}}{dz^2}=0$ at $\left(z=\frac{R}{2}\right)$

At z=R/2 the quantities in the parentheses of the denominator become the same, and the numerators cancel.
Thus, we have shown that the magnetic field at the midpoint of the Helmholtz coils is uniform



and has the value:

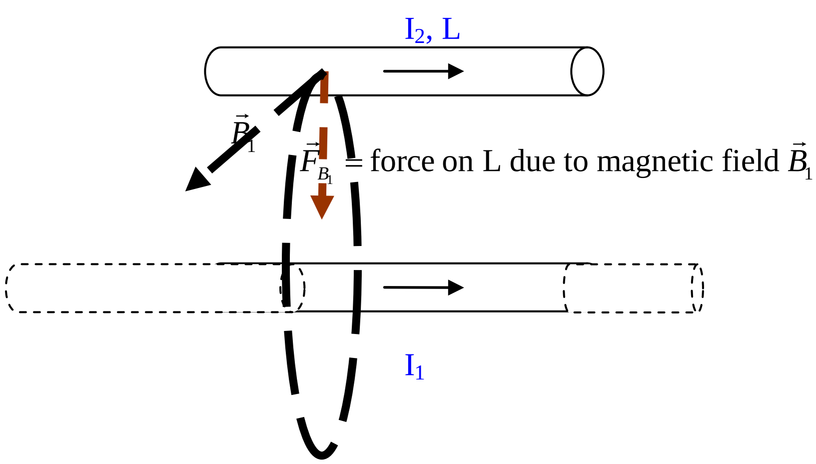
(30.31)

$$B=\frac{\mu_0 NI}{2}R^2\left(\frac{2}{\left(R^2+\frac{R^2}{4}\right)^{\frac{3}{2}}}\right)=\left(\frac{4}{5}\right)^{\frac{3}{2}}\frac{\mu_0 NIR^2}{R^3}=\left(\frac{4}{5}\right)^{\frac{3}{2}}\frac{\mu_0 NI}{R}$$

30.2 Magnetic Force Between 2 Conducting Parallel Wires :
If we place a conducting wire with current I_2 and length L into a magnetic field created by an infinitely long wire with current I_1, the wire L feels a magnetic force according to the earlier chapter 29.2

$$\int_{line} d\vec{F}_B = \int_{line} I\left(d\vec{s}\times\vec{B}\right) \Rightarrow I_2\vec{L}\times\vec{B}_1$$

If the magnetic field $\vec{B}_1$ in this equation comes from an infinitely long parallel wire with current I_1 : $B_1 = \frac{\mu_0}{2\pi}\frac{I_1}{r}$ one can easily see that the two wires will be attracted to each other, if they carry like charges moving in the same direction. They will be repelled if they move in opposite directions. **Both wires lie in the same plane of the page, one above the other**. The magnetic field of I_1 curls around the wire and points to the outside of the page (towards the viewer) at the wire lying above it. Use your right hand, with the thumb pointing in the direction of current I_1. The fingers of your right hand point in the direction of the magnetic field. Now, in order to determine the direction of the force on the wire, use your right hand again. The thumb points in the direction of I_2, your index finger points in the direction of the magnetic field $\vec{B}_1$, and your middle finger points downwards in the direction of the force.



(30.32) $\left|\vec{F}_{B_1}\right| = \left|I_2\vec{L}\times\vec{B}_1\right| = I_2 L B_1 = I_2 L\frac{\mu_0 I_1}{2\pi r}$

(According to Newton's third law the force of the magnetic field of I_2 on I_1 is equal and opposite in direction to the one just calculated. In the addendum of chapter 29, you can find the proof that Newton's third law is not true for currents which are not parallel or antiparallel to each other. Newton's laws apply strictly to classical, non-relativistic physics. Electromagnetism is actually a relativistic effect.) This formula remains correct even if the wire of I_2 is short.

It makes more sense to find the force between two wires per unit length:

(30.33) $$\boxed{\frac{F_B}{L} = \frac{\mu_0 I_1 I_2}{2\pi r}}$$

30.3 Ampere's law and Stokes' Theorem.

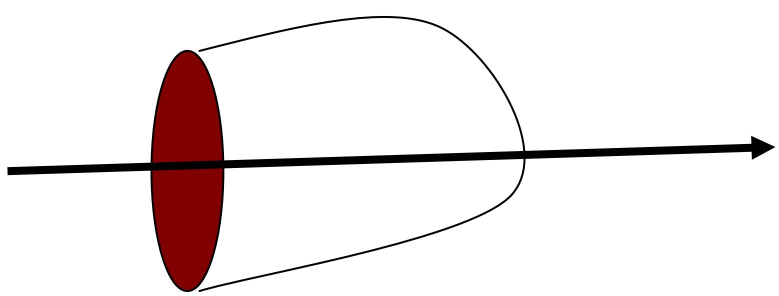
Whenever our situation displays a nice symmetry, there is another method for calculating the magnetic field, which is reminiscent of Gauss' law. It is called Ampere's law, which we have encountered in its local (differential) form (30.3), when we derived the law of Biot-Savart. It is the law that says that the magnetic field curls around the current density:

(30.34)
$$\boxed{curl\vec{B} = \mu_0 \vec{j}}$$

Whenever we have a current density $\vec{j}$ in a wire we arrive at the value for the current I itself by using the definition of the current as the flux of the current density through the cross-section of the wire carrying the current I, i.e. we perform a surface integral over a closed surface.

(30.35)
$$I = \iint_{surface\ \text{integral}} \vec{j}\cdot d\vec{A}$$

This surface is any surface **through which the current passes.** We only get contributions to the integral wherever the current density in not 0.



If, for example the wire has a circular shape of radius R, the current would be $I=jA=j\pi R^2$. If we apply the same integration to both sides of the differential form of Ampere's law, we face the task of having to integrate on the left side over $curl\vec{B}$, like here:

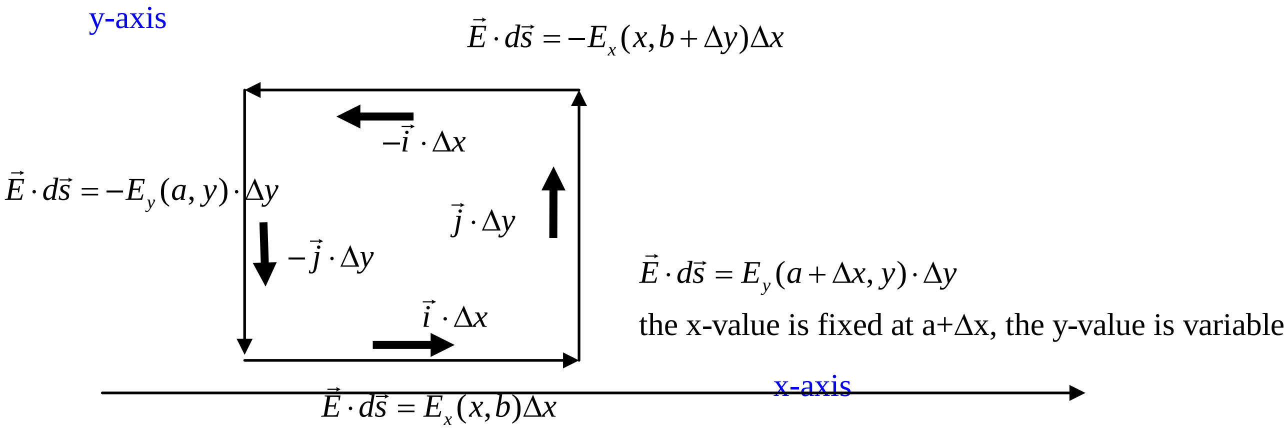
(3

Such surface integrals can be converted to path integrals over the boundary of the above surface (circulation) through *Stokes law* which states:

<u>30.3 a Stokes Law</u>: Quick and dirty proof:

Let us do this directly and calculate the circulation (line integral) of a vector field E around a rectangle with sides Δx and Δy. The lower left corner point of the square has the coordinates (a,b). We must calculate $E_x(x,y)\cdot dx$ along the horizontal lower and upper segments of the rectangle, and $E_y(x,y)\cdot dy$ along the vertical segments on the right and left of the rectangle.

$$\oint \vec{E}(x,y)\cdot d\vec{s} = \oint E_x(x,y)\cdot dx + E_y(x,y)\cdot dy$$



The total circulation is equal to:

(30.21) $$\oint \vec{E}\cdot d\vec{s} \approx E_x(x,b)\cdot\Delta x + E_y(a+\Delta x, y)\cdot\Delta y - E_x(x, b+\Delta y)\Delta x - E_y(a,y)\cdot\Delta y$$

Let us assemble the terms with the factor Δx and Δy respectively:

(30.22) $$\begin{aligned} E_x(x,b)\cdot\Delta x - E_x(x,b+\Delta y)\Delta x &= -\left[E_x(x,b+\Delta y) - E_x(x,b)\right]\Delta x \\ E_y(a+\Delta x, y)\cdot\Delta y - E_y(a,y)\cdot\Delta y &= \left[E_y(a+\Delta x, y) - E_y(a,y)\right]\Delta y \end{aligned}$$

We multiply the first expression by Δy/Δy, and the second by Δx/Δx:

(30.23) $$\frac{-\left[E_x(x,b+\Delta y) - E_x(x,b)\right]\Delta x\cdot\Delta y}{\Delta y}$$
$$\frac{\left[E_y(a+\Delta x, y) - E_y(a,y)\right]\Delta y\cdot\Delta x}{\Delta x}$$

In the limit both of these terms can be written as the respective partial derivatives:

(30.40) $$\frac{\left[E_y(a+\Delta x, y) - E_y(a,y)\right]}{\Delta x}\Delta x\Delta y \approx \frac{\partial E_y(a,y)}{\partial x}\Delta x\Delta y \Rightarrow\approx \frac{\partial E_y(x,y)}{\partial x}dxdy$$
$$-\frac{E_x(x,b+\Delta y) - E_x(x,b)}{\Delta y}\Delta x\Delta y \approx -\frac{\partial E_x(x,b)}{\partial y}\Delta x\Delta y \Rightarrow\approx -\frac{\partial E_x(x,y)}{\partial y}dxdy$$

Thus, the whole circulation becomes:

(30.41) $$\oint \vec{E}\cdot d\vec{s} \approx \frac{\partial E_y(x,y)}{\partial x}\Delta x\cdot\Delta y - \frac{\partial E_x(x,y)}{\partial y}\Delta y\cdot\Delta x$$

On the right hand side we have now the approximation of a double integral:

(30.42) $$\frac{\partial E_y(x,y)}{\partial x}\Delta x\cdot\Delta y - \frac{\partial E_x(x,y)}{\partial y}\Delta y\cdot\Delta x \approx \iint_A \left(\frac{\partial E_y(x,y)}{\partial x} - \frac{\partial E_x(x,y)}{\partial y}\right)dx\cdot dy$$

This is **Green's theorem**, which is nothing but the **third component of Stokes' theorem** in the plane: The circulation of any vector-field $\vec{E}$ around a closed loop is equal to the surface integral of $\partial_x E_y - \partial_y E_x$

Regular Stokes theorem in three dimensions:

The line integral of a vector field along a closed loop is equal to the surface integral of its curl: $\oint_{\partial A} \vec{E}\cdot d\vec{s} = \iint_A \vec{\nabla}\times\vec{E}\cdot d\vec{A}$. The surface is any surface which has the closed loop as its boundary.

It the vector field is a funtion of x and y only, the curl is reduced to the z-component.

The theorem is then called Green's theorem.

(30.43)
$$\oint_{\partial A} \vec{E}\cdot d\vec{s} = \iint_A \left(\underbrace{\frac{\partial E_y(x,y)}{\partial x} - \frac{\partial E_x(x,y)}{\partial y}}_{\left(\nabla\times\vec{E}\right)_Z} \right)\cdot dx\cdot dy$$

<u>Example</u>: Calculate the circulation of the vector field $\vec{B} = \langle 5y, -7x, 0\rangle$ around a rectangle with sides a and b:

<u>Answer</u>: $curl\vec{B} = \langle 0, 0, -12\rangle = -12\vec{k}$

The third component of this vector is parallel to the surface vector on the rectangle, and it is a constant. Therefore, the surface integral is the simple product $-12\vec{k}\iint_A dxdy\cdot\vec{k}$ = -12·a·b.

$\oint 5ydx - 7xdy = 5\cdot 0\cdot dx = -7ab - 5ba = -12ab;\ (0,0)\ to\ (a,0)\ to\ (a,b)\ to\ (0,b)\ to\ (0,0).$

The Green theorem (Stokes theorem in the plane) is usually stated in terms of a vector field with components P=E_x and Q=E_y; P(x,y) is the x-component, Q(x,y) is the y-component:

(30.44)
$$\oint_{\partial A} \vec{E}\cdot d\vec{s} = \oint_{\partial A} \langle P,Q\rangle\cdot d\vec{s} = \oint_{\partial A} Pdx + Qdy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cdot dx\cdot dy$$

If we choose Q=x and P=-y, the rhs becomes a surface integral:

(30.45)
$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cdot dx\cdot dy = 2\iint_A dx\cdot dy$$

With this choice we arrive at a way to calculate the surface of an area by means of the circulation around this area:

This circulation is equal to the surface integral of $curl\vec{B}$. But we know that $curl\vec{B} = \mu_0\vec{j}$ .The surface integral of $\mu_0\vec{j}$ is $\oiint_{A=circle} \mu_0\vec{j}\bullet d\vec{A} = \mu_0 I$. Therefore, we have $2\pi rB = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi r}$

(30.46)
$$\boxed{\oint_{\partial A} xdy - ydx = 2\iint_A dx\cdot dy}$$

For example, an ellipse can be written in parametric form as:

(30.47) $$x = a\cos\theta;\ y = b\sin\theta$$

Show that the area of an ellipse equals A=abπ

Example: Now, assume that we have a current density entering into the page. We know from earlier calculation that a magnetic field surrounds the current density. This means that the direction of the magnetic field around any loop, circling the current density is parallel to the tangential unit vector: $\vec{B} = B\vec{u}_\theta$ The circulation of this vector is given by:

(30.24) $$\oint_{circle} \vec{B}\cdot d\vec{s} = \oint_{circle} B\cdot r d\theta = 2\pi r B$$

End of intermission.

Stokes' theorem:

The flux of the curl of a vector field through any (simple) open surface is equal to the circulation of the vectorfield around a closed boundary of the surface. (In short: **Surface integral of the curl of the vector-function equals line-integral of the function= circulation**.) Think of the surface and the rim in analogy to a butterfly net and the rim that bounds it. The butterfly net can take on any shape, as long as it refers to the same rim**.**

(30.48) $$\iint_A \vec{\nabla}\times\vec{B}\cdot d\vec{A} \equiv \underbrace{\iint_A curl\vec{B}\bullet d\vec{A}}_{\substack{\text{the flux of curl}\vec{B}\\ \text{through any simple surface A}\\ \text{with the closed boundary }\partial A}} = \underbrace{\oint_{\partial A} \vec{B}\cdot d\vec{s}}_{\substack{\text{the circulation of }\vec{B}\\ \text{around the closed}\\ \text{boundary of any surface A}}}$$

This mathematical law of vector calculus is correct for any vector field (with the correct mathematical assumptions). The expression "circulation" is the anologue to flux. It is a line integral for a closed loop of any kind. It turns out that this law helps us to calculate the magnetic field very quickly in situations where there is a particular symmetry involved, e.g. cylindrical or spherical symmetry. In those situations the magnetic field is constant along the line of integration, and the integral becomes a simple product between the magnetic field and the length of the boundary line. We call the closed surface A and its boundary ∂A.

Just a reminder: As $div\vec{B} = 0$ always, the total or net flux of the magnetic field across any closed surface is always 0.

It is now easy to see how useful Stokes’ law is to calculate the magnetic field.
(Local or differential form of Ampere’s law.)

(30.49)
$$\boxed{curl\vec{B} = \mu_0 \vec{j}}$$

We see immediately that Stokes law can be used here :
(30.50)

$curl\vec{B} = \mu_0 \vec{j} \Rightarrow$ apply the surface integral (flux) through an open surface A to this equation:

$$\underbrace{\iint_A curl\vec{B} \bullet d\vec{A}}_{\substack{\text{the flux of curl}\vec{B} \\ \text{through any surface A}}} = \iint_{open\ surface\ A} \mu_0 \vec{i} \bullet d\vec{A} = \mu_0 I$$

$$\underbrace{\iint_A curl\vec{B} \bullet d\vec{A}}_{\substack{\text{the flux of curl}\vec{B} \\ \text{through any surface A} \\ \text{whose closed boundary is } \partial A}} = \underbrace{\oint_{\partial A} \vec{B} \cdot d\vec{s}}_{\substack{\text{the circulation of B} \\ \text{around the closed boundary} \\ \text{of the surface A}}} = \mu_0 I$$

In short, Ampere's law **states that** the line integral of a the magnetic field B around a loop through whose surface a current flows is proportional to that current**. (Remember that both the magnetic field and the current density are vector fields with vector components, and each component having spatial variables x, y, z)**

(30.51)

Ampere's law:
differential form: $curl\vec{B} = \mu_0 \vec{j}$
in integral form: $\oint_{\partial A} \vec{B} \cdot d\vec{s} = \mu_0 I = \mu_0 \iint_A \vec{j} \cdot d\vec{A}$

Let us apply this law also to the velocity vector field:
(30.52)
$$\vec{v} = \vec{\omega} \times \vec{r}$$

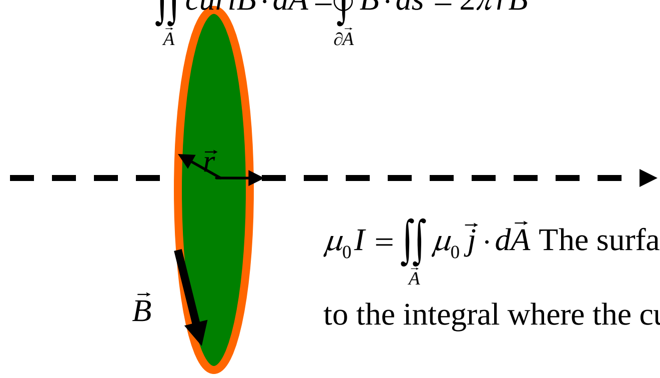
(30.53)
$$\iint_A curl\vec{v} \cdot \overrightarrow{dA} = \iint_A 2\vec{\omega} \cdot d\vec{A}$$
$$\oint \vec{v} \cdot d\vec{s} = v2\pi r = 2\omega\pi r^2 \Rightarrow v = \omega r$$

Examples for using Stokes' law in the calculation of the magnetic field:

30.3a Magnetic field created by a very long wire with current I.
If we apply this law to the calculation of a magnetic field created by a long wire we get very quickly the following result: At a distance r from the line of current we draw a circle with radius r. The current flowing through it is I. The line element ds is parallel to B and has the value rdθ. B is only dependent on r.
The integration gives simply:
(30.54)
$$2\pi r B = \mu_0 I$$

(30.55)
$$\boxed{B=\frac{\mu_0 I}{2\pi r}}$$

$$\iint\limits_{\vec{A}} curl\vec{B}\cdot d\vec{A}=\oint\limits_{\partial\vec{A}}\vec{B}\cdot d\vec{s}=2\pi rB$$



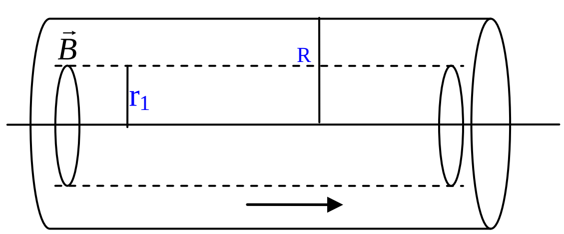
$\mu_0 I=\iint\limits_{\vec{A}}\mu_0\vec{j}\cdot d\vec{A}$ The surface A contributes only to the integral where the current density intercepts it.

(30.56)
$$\boxed{B=\frac{\mu_0}{2\pi}\frac{I}{r}}$$

This is of course the same result which we obtained when calculating this with Biot-Savart's law.

30.3b Magnetic field inside a wire with uniform current density j.

Now, if this wire is a cylinder, we can also easily calculate the magnetic field created by an inside portion of the current, i.e. the magnetic field inside the conducting cylindrical wire itself: We choose a cylindrical surface with radius r_1. The circulation of the magnetic field around the boundary of this surface is $2\pi r_1 B$ which is equal to $\mu_0 j\pi r_1^2$ Thus, the magnetic field between the two cylindrical surfaces is given by:

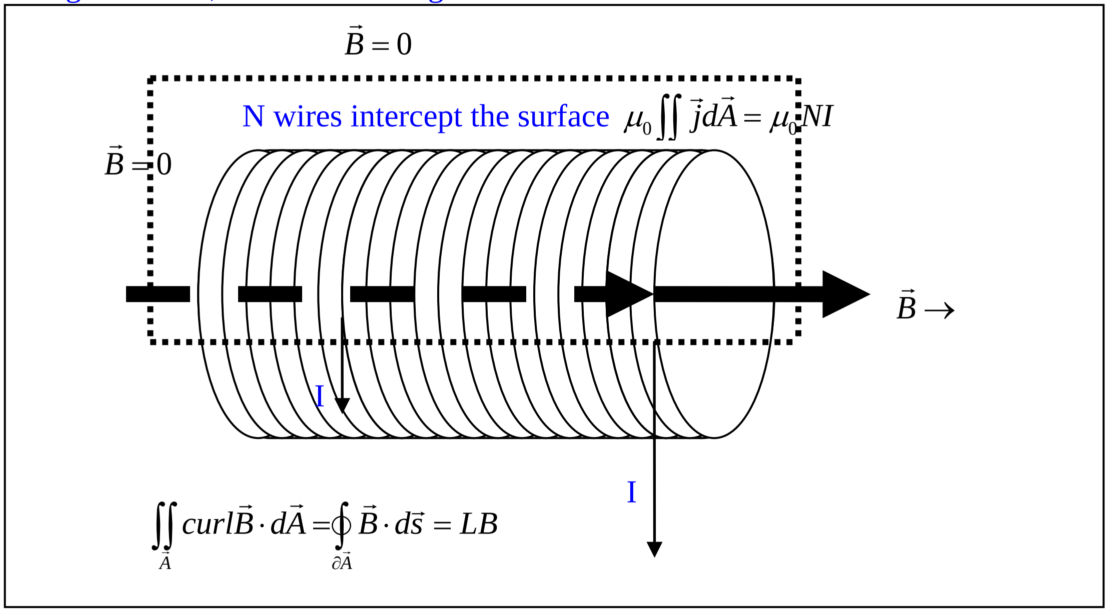


(30.25)
$$\boxed{B=\frac{\mu_0\pi r_1^2 j}{2\pi r_1}=\frac{\mu_0 j}{2}r_1;\ j=\frac{I}{A}=\frac{I}{\pi R^2}}$$

The magnetic field increases linearly with r_1 from the center of the wire to its boundary with maximum radius R.

30.3c The Magnetic Field of a Solenoid.

A solenoid is a long wire wound in a spiral fashion around an interior cylindrical space of length L. If this cylinder is long enough, there is a considerable stretch inside of the cylindrical space in which the magnetic field will be uniform. We know this from our calculation of the magnetic field at the center of a ring of current, see . There is a magnetic field at the center of any circular current, and this field is perpendicular to the surface of the loop, the z-axis in our example. If we put a great number of coils adjacent to each other, tightly wound , each of these loops will create a magnetic field, all of them along the central axis of the solenoid.



For an idealized situation we ignore the magnetic field outside the solenoid (it is weak), and get for the circulation of B simply BL. If there are N loops, N currents will intercept the rectangular area between the dashed lines; we get

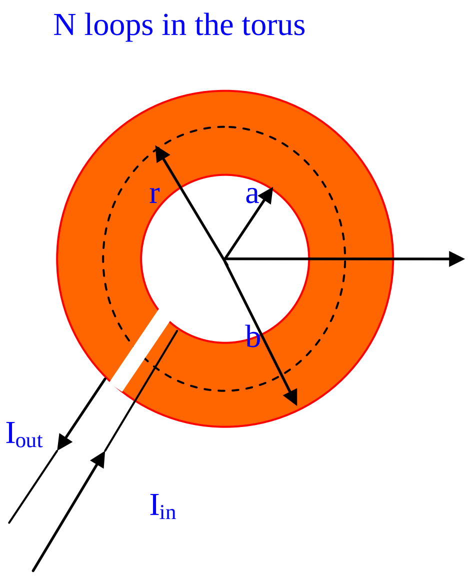
(30.58)
$$\boxed{B = \frac{\mu_0 NI}{L} = \mu_0 nI;\, n = \frac{N}{L}}$$

30.3d The magnetic field created by a toroid with N loops and radius r.

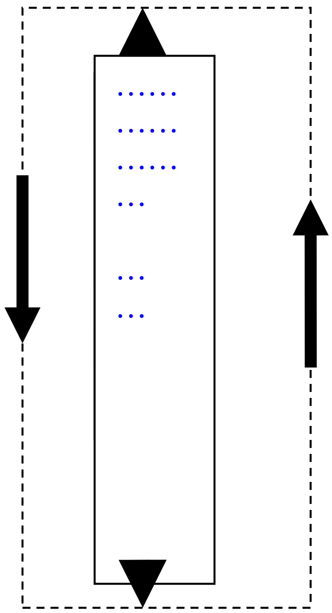
A toroid is a long solenoid bent into a circular shape. Let's calculate the magnetic field created by N loops inside of the toroid at a distance r from the central axis.Using Ampere's law, we see that the Stokes-Ampere surface (a circular sheet of radius R, concentric with the circular magnetic field) intercepts the current N times, whereas the circulation of the magnetic field inside is equal to the circumference of the circle times B:

$2\pi r B = \mu_0 NI \Rightarrow$

(30.59)
$$\boxed{B = \frac{\mu_0 NI}{2\pi r}}$$



30.3e Magnetic field of an infinite conducting sheet of current:
Assume that the cross-section of the sheet is perpendicular to the page, and the current is coming out of the sheet towards you. We draw a segment of this sheet:



The magnetic field tries to curl around the sheet and cannot do so without intercepting the current, which it cannot do either. The only thing left for it to do is to form a magnetic field parallel to this sheet on both sides of it. We choose as our surface the dashed Ampereian surface which is intercepted by the current. The total current flowing through this conducting sheet is equal to the current density times the cross-sectional area A=L·d. $I = jA = jLd$.

A magnetic field exists only along the vertical sides of length L. The circulation of L around the chosen rectangle is 0 on top and bottom and equal to BL on each side:

(30.26)
$$B2L = \mu_0 j\text{A} = \mu_0 j\text{Ld}$$
$$B = \frac{\mu_0 j\text{Ld}}{2L} = \frac{\mu_0 j\text{d}}{2}$$

(30.27) $$\boxed{B = \frac{\mu_0 jd}{2}}$$

<u>30.4 Gauss' law in Magnetism</u>.

As mentioned earlier, magnetism does not have any sources, or monopoles. Magnetic field lines close in themselves.This is the opposite to electrostatic field lines, which always emerge from a source (a positive charge) and disappear in a sink (a negative charge). This contrast is expressed by the two different differential formulas:

(30.28) $$\boxed{div\vec{E} = \frac{\rho}{\varepsilon_0} \text{ for electrostatic fields}}$$

and

(30.29) $$\boxed{div\vec{B} = 0 \text{ for all magnetic fields}}$$

In the case of the electric field we used the mathematical Gauss' law to calculate the electrostatic field in many situations with a simple geometry. We obtained Gauss' law for electrostatics by applying a volume integral to equation (30.28).

(30.30)
$$div\vec{E}=\frac{\rho}{\varepsilon_0}\Rightarrow \underbrace{\iiint_V div\vec{E}dV}_{\oint\!\!\oint_{\partial V}\vec{E}\cdot d\vec{A}}=\iiint_V\frac{\rho}{\varepsilon_0}dV=\frac{Q}{\varepsilon_0}$$

$$\oint\!\!\oint_{\partial V}\vec{E}\cdot d\vec{A}=\frac{Q}{\varepsilon_0}$$

The same approach leads to the surface integral of the magnetic field, for which we can also calculate the total flux through a close surface. This is always 0.
Note: distinguish carefully between the flux and the total flux through a closed surface. The latter is always 0 for a magnetic field.

(30.31)
$$\Phi_{B,total}=\oint\!\!\oint_{\partial V=A}\vec{B}\cdot d\vec{A}=0$$

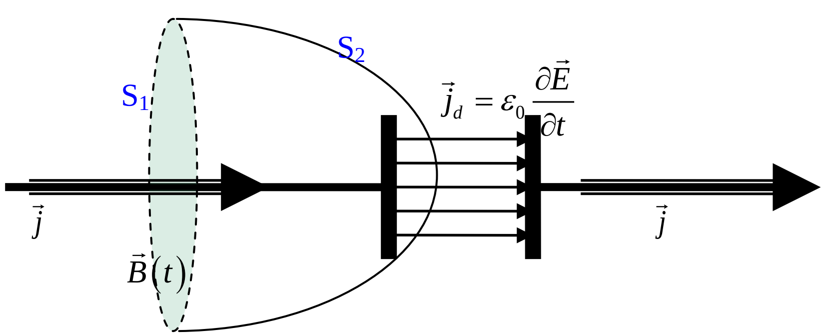
30.5 Displacement Current and the General Form of Ampere's Law.

The integral form of Ampere's law states that the circulation of a magnetic field around a closed curve equals the current intercepted by any surface which has the closed curve as a boundary.

Now, let us apply this information to a capacitor that is being charged or discharged. In this situation we know that there must be a changing electric field $\vec{E}(t)$ between the plates of the capacitor.
However, we can create two Amperian surfaces based on the same circulation around the conducting wire, one surface S_1 that intercepts the wire with its current, and another surface S_2 which intercepts the empty space between the capacitor plates, where we have the changing electric field. As we are getting the same magnetic field, we must also get the same "current value" in the empty space between the capacitors.
We know there are no charges flowing between the plates of the capacitor, but we also know that there is an electric field there, created by the charges on the capacitor plates.
The electric field between the plates of a capacitor is $E=\frac{\sigma}{\varepsilon_0}$ We get the total

$$\vec{E}(t)=\frac{\sigma}{\varepsilon_0};Q(t)=\sigma A=\varepsilon_0 A$$

$$I=\frac{dQ}{dt}=\varepsilon_0\frac{d\vec{E}}{dt}A=\varepsilon_0\frac{d}{dt}\iint_A\vec{E}\bullet d\vec{A}=\varepsilon_0\frac{d\Phi_E}{dt}$$



charge by multiplying E by the surface A, which is also the flux of E through the surface $\Phi_E = \oiint_A \vec{E} \cdot d\vec{A}$.

(30.31) $$Q = \sigma A; E = \frac{\sigma}{\varepsilon_0} \Rightarrow \sigma = E\varepsilon_0 \Rightarrow Q = \varepsilon_0 EA = \varepsilon_0 \iint_A \vec{E} \cdot d\vec{A}$$

To get an equivalent current, designated as displacement current I_d, we take the derivative of Q with respect to time.

(30.32) $$\boxed{\frac{dQ}{dt} = I_d = \varepsilon_0 \frac{d}{dt} \underbrace{\iint_A \vec{E} \cdot d\vec{A}}_{\Phi_E} = \varepsilon_0 \frac{d\Phi_E}{dt}}$$

Equation (30.32) represents the integral form of this situation, after integration over the surface. By dividing the equation by the surface area A we get the local form.

(30.33) $$I_d = \iint_A \vec{j}_d \cdot d\vec{A}; \Phi_E = \iint_A \vec{E} \cdot d\vec{A}$$

(30.34) $$\boxed{\vec{j}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

Thus we have discovered the concept of a deplacement current, and its density, which must be exactly equal in value to the current and current density in the wire, respectively.

(30.35) $$\vec{j}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \text{ and } I_d = \iint_{surface} \vec{j}_d d\vec{A} = \varepsilon_0 \frac{d}{dt} \underbrace{\iint_{surface} \vec{E} d\vec{A}}_{\text{flux of } \vec{E} \text{ through } \vec{A}} = \varepsilon_0 \iint_{surface} \frac{\partial \vec{E}}{\partial t} d\vec{A}$$

(30.36) $$I_d = \varepsilon_0 \frac{d\Phi_E}{dt}$$

We realize that a magnetic field cannot only be created by a current in a wire, but also by a time changing electric field $\vec{E}(t)$ as it appears between the plates of a capacitor being charged or discharged. The displacement current density must be added to the right side of Ampere's law as $\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}_d$.

(30.37) $$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \underbrace{\mu_0 \, \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}_{\vec{j}_d}$$

This gives us the complete form of the so-called Ampere-Maxwell law:

(30.38)
$$\vec{\nabla}\times\vec{B}=\mu_0\vec{j}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$
$$\underset{\substack{\text{boundary of}\\ \text{surface A}}}{\oint}\vec{B}d\vec{s}=\mu_0 I+\mu_0\varepsilon_0\underbrace{\frac{d\Phi_E}{dt}}_{\text{flux of } E}$$

When there is no regular current but just a time changing electric field we still get a magnetic field, which, not surprisingly, will also change in time:

(30.38)
$$\vec{\nabla}\times\vec{B}=\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

Time changing electric fields $\vec{E}(r,t)$ create magnetic fields $\vec{B}(r,t)$ around them.

30.6 Magnetism in Matter (Optional).

According to the classical Bohr model for atoms electrons are in orbit around the nucleus. Thus, they create a magnetic field inside them, because they constitute, in effect, magnetic dipole moments.

In chapter 29 we defined the dipole moment of a current as the product between the current of the loop times the area of the loop, in the direction of the normal to the surface of this loop:

(30.38) $$\vec{\mu}=I\vec{A}$$

We calculated the magnetic field of a dipole earlier in $\boxed{B=\frac{\mu_0 I}{2}\frac{R^2}{z^3}}$ (long distance from the center of the coil.) and $B=\frac{\mu_0 I}{2R}=\frac{\mu_0 q}{2RT}$ at the center of the loop.

This is a current loop (a loop of moving charges) which will experience a torque $\vec{\tau}=\vec{\mu}\times\vec{B}$ when placed into another magnetic field, and will consequently involve a potential energy $U=-\vec{\mu}\cdot\vec{B}$

Now, we can consider any electron in orbit around its nucleus as such a current loop. Let us calculate its physical quantities related to its electric charge. The current consists of a single electron, circulating around the nucleus in the time T, where T is the period. It also has an angular momentum $L=m\text{v}R$, which points in the opposite direction of the magnetic moment.

(30.38) $I=\frac{dQ}{dt}=\frac{e}{T}=\frac{e\text{v}}{2\pi R}$; $\text{v}=\frac{2\pi R}{T}$

It is convenient to express the magnetic moment in terms of the angular momentum L:

(30.38) $\mu = I \cdot A = \frac{ev}{2\pi R}\pi R^2 = \frac{1}{2}evR = \left(\frac{e}{2m}\right)L$

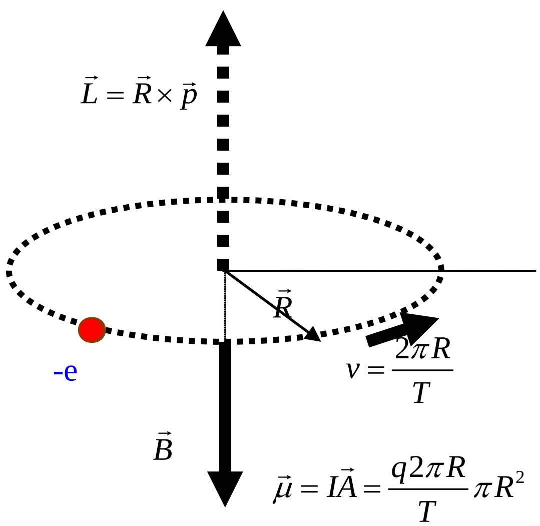
angular momentum L=mvR

(30.38) $L = \hbar\sqrt{l(l+1)};\ l = 1,2,3...$

for l =1 we get the smallest magnetic moment of an **orbiting** electron

(30.38) $\mu = \left(\frac{e}{2m}\right)L = \left(\frac{e}{2m}\right)\sqrt{2}\cdot\hbar$

The magnetic moment of the orbital electron points in the opposite direction of the angular momentum, because of the negative charge for the electron.
In the Bohr model of the atom we assumed that the angular momentum of the electron was equal to a multiple of $\hbar$, $L = n\hbar$. This was the beginning of quantum theory which ultimately proved that the angular momentum is rather given by the formula (30.38) $L = \hbar\sqrt{l(l+1)};\ l = 1,2,3...$
In this formula l = 1,2,3.. is called the orbital quantum number, an positive integer value which relates to the orbital angular momentum of the electron.
The electron all by itself has a so-called spin quantum number of ½. The spin can be visualized or imagined as a rotation of the electron or proton around their own respective axes. And its spin angular momentum is given by: (30.38) $L_S = \hbar\sqrt{s(s+1)} = \hbar\frac{\sqrt{3}}{2}$
The spin magnetic moment of of an electron, called the Bohr magneton, turns out to be:



(30.38)

$\mu_S = \frac{e\hbar}{2m_e} = 9.27\cdot 10^{-24}\frac{J}{T} = \text{Bohr magneton } \mu_B$

Atomic magnetic moments are often expressed as multiples of the Bohr magneton, which is a kind of simple unit for those values.

Summary of permanent magnets:
Some metals display **permanent magnetism**, and are called **ferro magnetic**, due to permanent circular domains of currents inside the metallic structure. This magnetism can be enhanced by placing the magnet into a magnetic field. In turn, the **magnetic field of**

the ferro magnetic enhances the original magnetic field. The magnetic state of a substance is described by a quantity $\vec{M}$ the magnetization vector. **The <u>magnetization vector is the magnetic moment per unit volume of the substance</u>**.

When a substance is placed into an external magnetic field $\vec{B}_0$ of a current carrying conductor, the total magnetic field $\vec{B}$ is a combination of the external field $\vec{B}_0$ and an additional magnetic field $\vec{B}_m$ due to magnetic moments of atoms and electrons within the substance.

(30.39)
$$\vec{B} = \underbrace{\vec{B}_0}_{\substack{\text{Magnetic field} \\ \text{of the vacuum}}} + \underbrace{\vec{B}_m}_{\substack{\text{magnetic field} \\ \text{of the magnetic substance}}}$$

We express B_m through the magnetization vector M. Let us see how we are being let to the notion of M as the magnetization vector. Let us for that purpose express the magnetic field of a solenoid with N loops and length h, in terms of the magnetic moment $\vec{\mu} = NI\vec{A}$ and the volume containting the magnetic field. This volume would be the volume of the magnetic substance.

(30.40)
$$B_m = \mu_0 nI = \mu_0 \frac{N}{h} I; \; \mu = NIA \Rightarrow NI = \frac{\mu}{A}$$
$$B_m = \mu_0 \frac{\mu}{hA} = \mu_0 \frac{\mu}{V} = \mu_0 M$$

where $\vec{M}$ is the magnetization vector. The magnetization vector is the magnetic moment per unit volume of the substance. The total magnetic field B at a point within a substance depends on both the applied field B_0 and the magnetization of the substance.
Let us see how we can arrive at the concept of magnetic moment per unit volume:
We start with the magnetic field of a long solenoid:

A is the surface of the magnetic moment; N =turns of wire of the solenoid.

(30.41) V is the volume inside the solenoid.

$$\text{M=magnetization=}\frac{\text{magnetic moment of the magnetized material}}{\text{volume of the magnetized material}}$$

(30.41)
$$\boxed{\vec{B} = \vec{B}_0 + \mu_0 \vec{M}}$$

B_0 is the magnetic field of the vacuum, due to macroscopic currents as discussed. When this field permeates a magnetic substance, or a substance that can be magnetized, it is convenient to express this field also in terms of the magnetic moment per unit volume involved. So, we introduce yet another quantity H, which is the magnetic moment due to currents. It is called magnetic field strength. The magnetic field B (not B_0) on the other hand is called the magnetic flux density or the magnetic induction. This magnetic induction can then be written as:

(30.42)
$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right) = \vec{B}_m + \mu_0 \vec{M}$$

The quantities magnetic field strength and magnetization, $\vec{H}$ and $\vec{M}$,**have the same** units of Ampere/m:

(30.43) $$[H]=[M]=\frac{[\text{magnetic moment}]}{[volume]}=\frac{[I\vec{A}]}{[V]}=\frac{Amp\cdot m^2}{m^3}=\frac{Amp}{m}$$

For a solenoid in the vacuum with the current turnt on we have no magnetization and get:

(30.44) $$B=\mu_0(H+\cancel{M})=B_0=\mu_0 H=\mu_0 nI \\ H=nI$$

When we place demagnetized iron into the cavity of a solenoid (without current) the original magnetic field as well as H are 0. If we turn the current on, both B and H increase change. H increases with the current, but M, the magnetization of the substance also increases. The resulting magnetic field B must now be considered:

(30.45) $$\vec{B}=\mu_0\left(\vec{H}+\vec{M}\right)$$

Materials can be classified as to their behavior in an exterior magnetic field. Some become permanent magnets. They maintain a net magnetic field expressed by the magnetization vector M. They are the paramagnetic and ferro-magnetic materials. Material in which the induced magnetic moments become random after the material is removed from the exterior magnetic field is called diamagnetic.

The magnetization M of paramagnetic and ferromagnetic substances is proportional to the magnetic field H brought about by exterior currents. Such substances are susceptible to be magnetized, hence the name for the proportionality factor $\chi\,(chi=kye)$

(30.46) $$\vec{M}=\chi\vec{H};\vec{B}_0=\mu_0\vec{H}\Leftrightarrow\vec{H}=\frac{\vec{B}_0}{\mu_0}$$

where $\chi\,(chi=kye)$ is the magnetic susceptibility.

In both paramagnetic and ferromagnet substances the positive. In ferromagnetic substances, interactions between the atoms cause magnetic moments to align and create a strong magnetization that remains after the external field is removed. In diamagnetic substances susceptibility is negative.

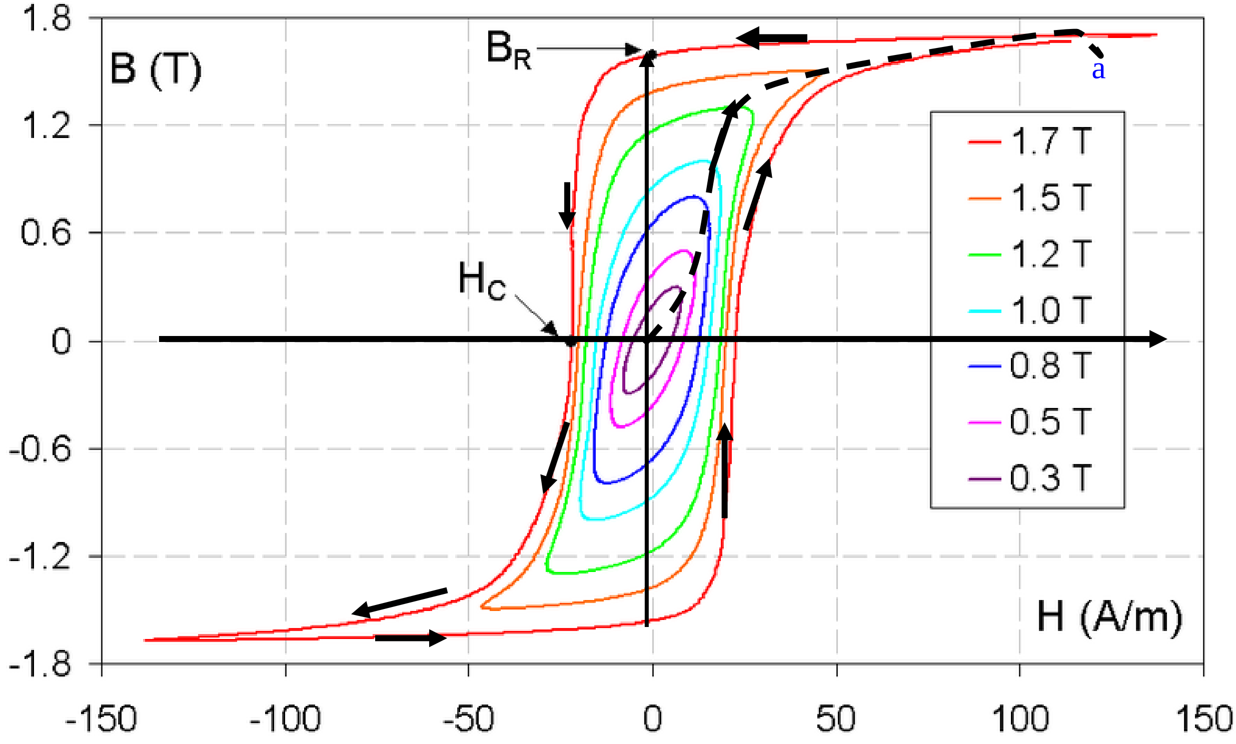
We summarize:

(30.47) $$\vec{B}=\mu_0\left(\vec{H}+\vec{M}\right)=\mu_0\left(\vec{H}+\chi\vec{H}\right)=\mu_0(1+\chi)\vec{H}=\mu_m\vec{H}$$

(30.48) $$\vec{B}=\mu_m\vec{H};\mu_m=\text{magnetic permeability} \\ \mu_m=\mu_0(1+\chi);\mu_0\text{ permeability of the vacuum}$$

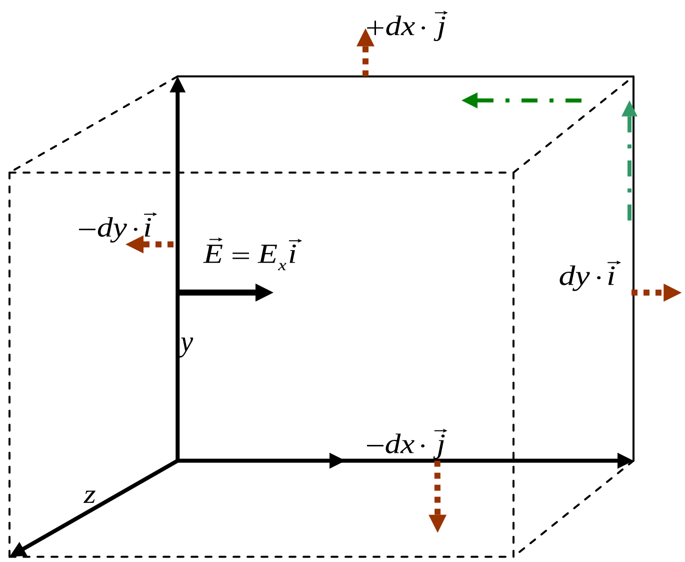
For paramagnetic and ferromagnetic substances the magnetic permeability is larger than the permeability of the vaccuum, for diamagnetic material it is smaller. For ferromagnetic material which is used to strengthen the magnetic field inside of coils the magnetic permeabiltiy can be 5000 times the permeability of the vacuum.

When ferromagnetic substances are placed into a magnetic field and the current is being increased, the resulting magnetic field B, plotted versus the magnetic field H due to the currents of the coil, shows the behavior below which is called a hysteresis curve.



The domains with their magnetic moments inside the iron get lined up to increase the total magnetic field. When the current is turned off, the magnetized domains remain aligned and form a permanent magnet.

Appendix (Optional)**:**
Derivation of Stokes' law, just for fun.



We start by drawing a cube to which we apply Gauss's law. Then we use this law and apply it to the two dimensional projection onto the x-y plane:

(30.49) $$\iiint\limits_{volume} div\vec{E}dV = \oiint\limits_{surface} \vec{E}d\vec{A}$$

We project this law onto the x-y plane, the volume element becomes a surface element dxdy, the surface element vector $d\vec{A}$ perpendicular to the surface of volume V becomes $\vec{n}$dl which is a line element perpendicular to boundary of the projected surface.

(30.50)
$\vec{n}dl \equiv d\vec{n} = \langle dy, -dx \rangle$ is the normal line element to a curve in the x-y plane
$$\vec{E}\cdot\vec{n}dl = E_x dy - E_y dx$$

(30.51) Compare this to the tangential line element:
$$d\vec{s} = \langle dx, dy \rangle$$

The scalar product between the tangential and the normal line elements must be 0.

(30.51) $$\langle dx, dy \rangle \bullet \langle dy, -dx \rangle = 0$$

(30.51) $$d\vec{s}\times d\vec{n} = -\vec{i}$$

We choose the orientation of $d\vec{n}$ such that it points to the outside of the closed curve traversed by the vector $d\vec{s}$ such that the area lies to the left of the direction of $d\vec{s}$ The flux of the one dimensional electric field $\vec{E} = E_x(x,y)\vec{i}$ through the infinitesimal surface becomes

$$\vec{E}\cdot d\vec{n} = E_x(x)\vec{\mathrm{i}}\cdot\left(-dy\vec{\mathrm{i}}\right) + E_x\left(x+dx\right)\vec{\mathrm{i}}\cdot dy\vec{\mathrm{i}} = \left[E_x\left(x+dx\right) - E_x(x)\right]dy =$$

in the xy plane $$\frac{\left[E_x\left(x+dx\right) - E_x(x)\right]}{dx}dydx = \frac{\partial E_x}{\partial x}dydx$$

We recognize this again as the first term of the divergence of $\vec{E} = \langle E_x, E_y \rangle$ multiplied with the two dimensional volume element $dxdy$

(30.52) $$\iint_{\text{rectangle}} div\vec{E} \cdot dxdy = \iint_{\text{rectangle}} \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} \right) \cdot dxdy = \oint_{\substack{\text{boundary of}\\ \text{the rectangle}}} E_x dy - E_y dx$$

The projection of Stokes law onto the x-y plane yields :

(30.53) $$\oiint_{\substack{\text{the flux of curlB}\\ \text{through any surface A}}} curl\vec{B} \bullet d\vec{A} = \oint_{\substack{\text{the circulation of B}\\ \text{around the boundary}\\ \text{of the surface A}}} \vec{B} \cdot d\vec{s}$$ becomes, when projected onto the xy plane:

$$\oiint_{\substack{\text{the flux of curlB}\\ \text{through the projection of}\\ \text{surface A on xy plane}}} \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \vec{k} \bullet dxdy\vec{k} = \oiint_{\substack{\text{the flux of curlB}\\ \text{through the projection of}\\ \text{surface A on xy plane}}} \left(curl\vec{B} \right)_z \bullet dxdy = \iint_{A(x,y)} \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \bullet dxdy$$

$$\oint_{\substack{\text{the circulation of B}\\ \text{around the boundary}\\ \text{of the projected}\\ \text{surface A(x,y)}}} \vec{B} \cdot d\vec{s} = \oint_{\substack{\text{the circulation of B}\\ \text{around the boundary}\\ \text{of the projected}\\ \text{surface A(x,y)}}} B_x dx + B_y dy$$

To summarize the essential formulas in this last calculation:

(30.54) $$\iint_{A(x,y)} \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \bullet dxdy = \oint_{\substack{\text{the circulation of B}\\ \text{around the boundary}\\ \text{of the projected}\\ \text{surface A(x,y)}}} B_x dx + B_y dy$$

If we compare the two formulas (30.52) and (30.54) we see that they are identical under the substitution of :

(30.55) $$E_x \to B_y \text{ and } E_y \to -B_x$$

Thus, **<u>Gauss' law for the plane can be considered as the third component (z-component) of Stokes' law</u>**.

Summary of Gauss and Stokes's laws**:**

Gauss law in space:

(30.56) $$\boxed{\iiint_V div\vec{E} dV = \oiint_{\partial V} \vec{E} \bullet d\vec{A}}$$

Gauss law in the x-y plane:

(30.57) $$\iint_A div\vec{E} \cdot dx \cdot dy = \oint_{\partial A} E_x dy - E_y dx = \int_{\partial A} \vec{E} \bullet \vec{n} dl; \vec{n} dl = \vec{i} dy - \vec{j} dx$$

Stokes Theorem for magnetic fields**:**

(30.58) $$\boxed{\iint_A curl\vec{B} \bullet d\vec{A} = \oint_{\partial A} \vec{B} \bullet d\vec{s}}$$

Examples:

In (30.57), if $div\vec{E} = \text{constant}$, the integral is proportional to the area A. For example

(30.59) $$\vec{E} = \langle x, y \rangle ;\, div\vec{E} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1 + 1 = 2$$

(30.60) $$\iint_A div\vec{E} \cdot dx \cdot dy = 2 \underbrace{\iint_A dxdy}_{\text{area of the surface A}} = \oint_{\partial A} xdy - ydx$$

Thus we have found a new way to calculate the area of a plane surface A, by calculating the line integral around it:

(30.61) $$\boxed{\iint_A dxdy = \frac{1}{2} \oint_{\partial A} xdy - ydx}$$