Homework: See website.

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29.1 Magnetic Fields and Forces.

We all know how bar magnets attract some metallic objects like iron files. Many experiments have revealed that magnetic field lines always start in one pole and end up in another. There are no magnetic monopoles out of which magnetic fields arise. This is in exact contrast to electric fields for which there are positive or negative charges. (The magnetic fields generated by magnets have their origin in circular currents inside of the so-called para-magnetic material.) Let us approach the concept of magnetic fields by contrasting them to electric fields:

The mathematical description for electric fields emerging from single charges was:

(29.1) $$div\vec{E} = \frac{\rho}{\varepsilon_0}$$

Applying a volume integral and using Gauss' theorem this leads to the Gaussian law:

(29.2) $$\iiint\limits_{volume} div\vec{E}dV = \oiint\limits_{\substack{surface\\ \text{of the volume}}} \vec{E}d\vec{A} = \frac{Q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\text{charge inside the volume}$$

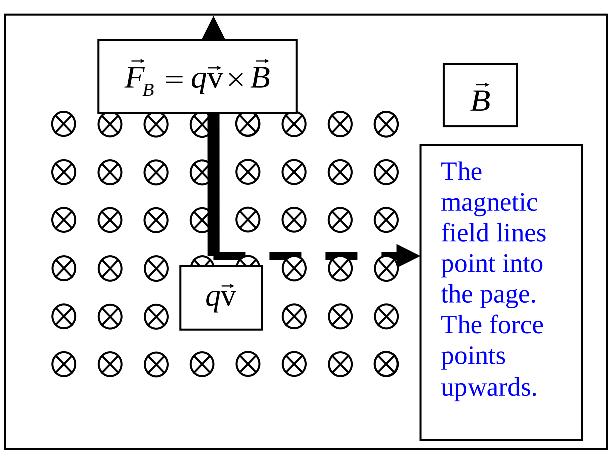
For a magnetic field B we have always :

(29.3) $$\boxed{div\vec{B} = 0}$$

This is the same as saying that magnetic fields do not have sources or sinks or monopoles. Magnetic field lines appear between the poles of a permanent magnet. One is called the Southpole, the other the Northpole. By convention, we say that the magnetic fieldlines are directed from the Northpole to the Southpole. No matter how many times we cut a bar magnet in half, we always end up with two poles, which attract other magnets. The Southpole of one magnet is attracted by the Northpole of another magnet, and vice versa. We have found in the last century that all permanent magnets are due to many circular little currents inside of the magnetic material. The magnetic field of the earth is due to a huge circular current of molten iron inside the earth. This current, and with it the magnetic field of the earth, has changed during the geological history of the Earth. The magnetic poles do not coincide perfectly with the geographic poles. And actually, the magnetic Southpole corresponds roughly to the geographic Northpole. The N-point of the compass needle points to the magnetic Southpole of the Earth, which is the geographic Northpole.

It is easy to observe that any **moving charge q** is deflected when entering a magnetic field according to:

(29.4) $$\boxed{\vec{F}_B = q\vec{v}\times\vec{B}}$$



From this we can deduce the unit for the magnetic field, which is

(29.5) $$[B] = \frac{Newtons \cdot s}{Coulombs \cdot m} = 1tesla = 1T = 10^4 Gauss$$

The force is of course present in addition to the force created by an electric field. The total force on a charge q is therefore:

(29.6) $$\boxed{\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \text{ Lorentz Force}}$$

The work done by the force of a **uniform** magnetic field (not time dependent) on a moving charge is always 0 as we can easily see (A time-dependent magnetic field creates an electric field, see later):

(29.7) $$dW = \vec{F}\cdot d\vec{s} = q\left(\vec{v}\times\vec{B}\right)\cdot d\vec{s} = 0$$

$\vec{v}$ is parallel to $d\vec{s}$

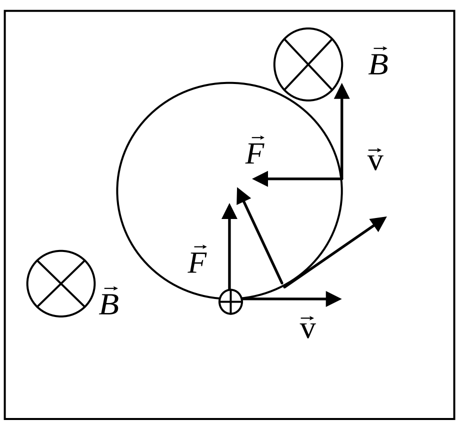
The force is perpendicular to both the velocity and the magnetic field vector. $d\vec{s} \parallel \vec{v} \Rightarrow d\vec{s} \perp \vec{F}$, similar to the gravitational force on a planet in orbit, which also does not do any work because the force is perpendicular to the velocity. From the Work-Energy theorem we know that

(29.8) $$W = \Delta K = K_2 - K_1 = \Delta\left(\frac{1}{2}mv^2\right)$$

As work is 0, there can be no change in the kinetic energy of a charge in a magnetic field. The direction of the velocity of a charged particle in a magnetic field can change, but not its magnitude, or its kinetic energy.

<u>29.2 Motion of a Charged Particle in a Uniform Magnetic Field</u>.

29.2a Movement perpendicular to the magnetic field; Cyclotron frequency:



Assume that we have a positively charged particle, like a proton, injected into a uniform magnetic field such that the **<u>initial velocity of the particle is perpendicular to the field</u>**. Assume that the $\vec{B}$ field points into the plane: The charged particle will experience a force perpendicular to both the velocity and the magnetic field. It experiences a centripetal acceleration which causes it to move in a circle:

$$\vec{F} = q\vec{v}\times\vec{B} = m\vec{a}$$

(29.9)

$$q\text{vB} = \frac{\text{mv}^2}{\text{r}} = m\omega^2 r;\ \text{v} = \frac{\text{qBr}}{\text{m}};\ r = \frac{m\text{v}}{qB}$$

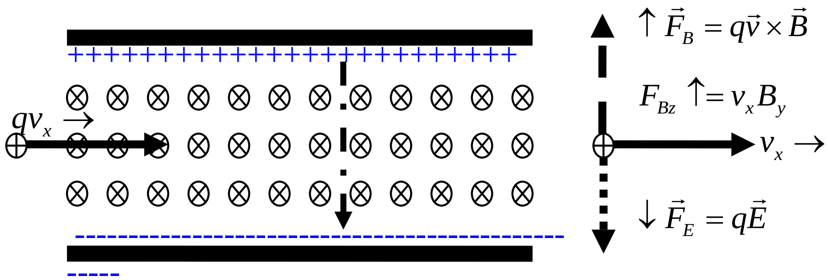
(29.10) $$\boxed{\omega = \frac{qB}{m} = \text{cyclotron frequency}}$$

29.3 Applications.

29.3a Velocity Selector :

We can use the set-up of perpendiculary **crossed electric and magnetic fields** to select exact velocities out of a particle beam. In the picture below the magnetic field points into the plane. We place a parallel plate capacitor inside a uniform magnetic field. The particle beam with varying velocities points from the left and enters the capacitor. The electric field points from top to bottom. Only those particles will pass through the crossed field area for which the upward magnetic force equals the downward electric force, resulting in particles with the speed equal to

$$v = \frac{E}{B}$$



We have to cross them in such a way that the resultant Lorenz force on the particle is 0.

$$-q\vec{E} = q\vec{v} \times \vec{B}$$

$$Choose: \mathrm{v}_x, B_y \Rightarrow q\vec{v} \times \vec{B} = q\mathrm{v}_x B_y \vec{k}$$

(29.11)
$$Choose: \vec{E} = -E_z \vec{k} \Rightarrow q\vec{E} = -qE_z \vec{k}$$

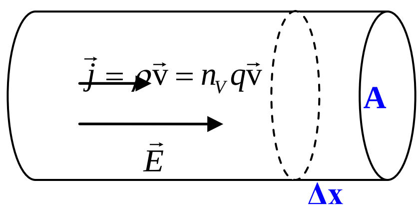
$$qE_z = q\mathrm{v}_x B_y \Rightarrow \qquad \mathrm{v}_x = \frac{\mathrm{E}_z}{\mathrm{B}_y}$$

Only particles with this exact velocity will continue in a straight line, all others will be deflected up or down and hit the capacitor plates. If we inject these particles with an exactly known velocity into a perpendicular magnetic field, the particles will be deflected into a circular motion according to (29.9). By measuring the radius of their circular motion the ratio between mass and charge can be exactly determined. (Thomson's e/m experiment).

We see here also that the ratio between the electric and the magnectic field strength has the dimension of a velocity : m/s (We shall see later that in an electromagnetic field this ratio is equal to the speed of light c.)

29.4 Force on a current carrying conducting wire in a magnetic field:

If we insert a conducting wire into a magnetic field B, this field will obviously excert a force on each one of the conducting electrons or charges moving with the drift velocity v_d through this wire. The total number of charges in a **cylindrical segment of wire with length L and crosssection A** is simple to calculate:



It is equal to the charge density times the volume of the cylinder $\rho_q A \cdot L = \frac{N}{V} q \cdot A \cdot L$. We get an infinitesimal force dF acting on a small portion of the wire by using just a small distance for the length, i.e. Δx. When a current I is flowing with the speed v=dx/dt the infinitesimal amount of charges affected in a small segment of wire is given by $\rho_q \cdot A \cdot dx = \rho_q A v dt$. Now we recognize that this expression is equal to the total current I times dt:

(29.12) $\underbrace{\rho_q v}_{j} A\, dt = \vec{j} \cdot \vec{A} \cdot dt = I \cdot dt = dQ$

In the formula for the force on a single charge q we have to transform this expression for a single charge into the expression for a small amount of charges dQ passing through a segment of wire with cross section A and length dx.

(29.13)
$$dQ = I \cdot dt$$
$$d\vec{F} = dQ \cdot \vec{v} \times \vec{B} = I \cdot dt \cdot \vec{v} \times \vec{B}$$

We can write the velocity vector as a product of its magnitude and a unit vector. We then cancel the velocity magnitude in the numerator with the denominator.

(29.14)
$$d\vec{F} = I \cdot \frac{ds}{v} \cdot \vec{v} \times \vec{B} = I \cdot \frac{ds}{v} \cdot v \cdot \vec{u} \times \vec{B}$$

$ds \cdot \vec{u} = d\vec{s}$ is a small segment of the current carrying wire in the direction of the wire, which is the direction of the current density: That leaves us with the product

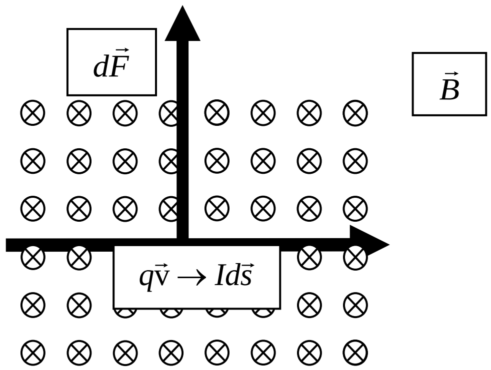
(29.15)
$$\boxed{d\vec{F}_B = I ds \cdot \vec{u} \times \vec{B} = I d\vec{s} \times \vec{B}}$$

Caution with the definitions here:

(29.16)
$N =$ number of charges
q=individual elementary charge= $\pm 1.60 \cdot 10^{-19} C$
Q=Nq=total charge; $n_V =$ number density of charges
dQ=Idt = small amount of charges flowing through a segment of wire
$n_V q\vec{v} =$ current density= $\vec{j}$; in our model: Volume V=A $\cdot \Delta x$; I=j $\cdot$ A= $\frac{dQ}{dt} = \iint_A \vec{j} \cdot d\vec{A}$

For a small portion of the wire we get the infinitesimal force :

(29.16) $\boxed{d\vec{F}_B = I\left(d\vec{s} \times \vec{B}\right)}$

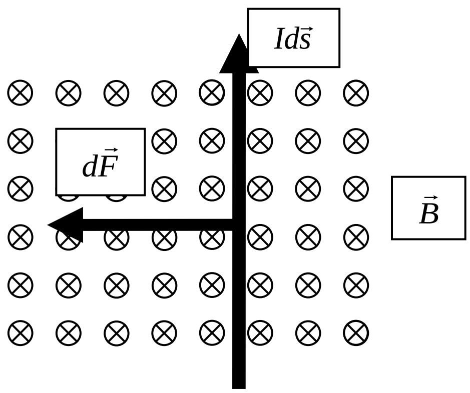


$d\vec{F} = Id\vec{s} \times \vec{B}$ becomes for a straight wire of length L:

(29.17) $\vec{F} = I\left(\vec{L} \times \vec{B}\right)$

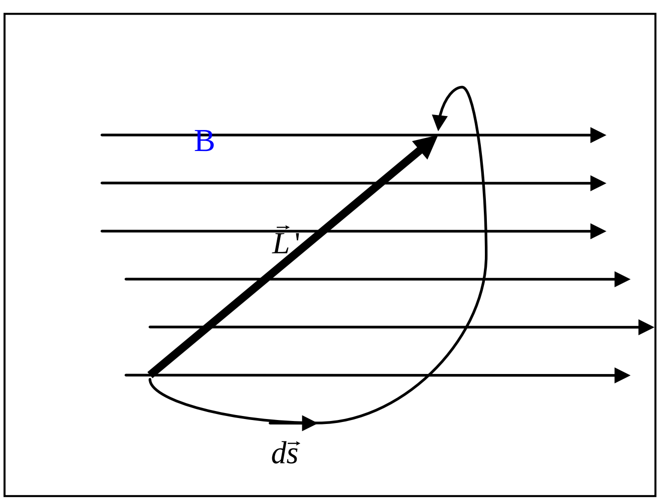
In general, we must integrate over the whole length of the current carrying wire to get the total force exerted on this wire:

(29.18) $\int_{line} d\vec{F}_B = \int_{line} I\left(d\vec{s} \times \vec{B}\right)$



Note the direction of the force : It is perpendicular to both the line-segment and to the magnetic field lines. Whenever the current and the magnetic field are in the same plane, the force is perpendicular to that plane. (Review the “right hand rule” which is valid for all cross products, of course.)

Here, to the left, the current is directed upward, the magnetic field is directed into the page, the force is directed to the left. A flexible wire will be bent to the left.



If the field is **uniform (constant)**, the integration is just over the line and is nothing but the **vector sum over infinitesimal segments making up the line**. The vector sume connects the tail of the first vector segment with the head of the last vector segment. This vector sum is therefore equal to the **straight line vector** connecting the initial point of the wire to its endpoint.

This sum is 0 if the line forms a closed loop. (Like the total velocity or displacement vector for a closed path.) To summarize:

The magnetic force on a curved current carrying wire in a **uniform** magnetic field is equal to that on a straight wire, connecting the endpoints of this wire.

(29.19)
$$\vec{F}_B = \left(I \int_{line} d\vec{s} \right) \times \vec{B} = I \int_a^b d\vec{s} \times \vec{B} = I\vec{L}' \times \vec{B}$$
$$\int_a^b d\vec{s} = L'$$

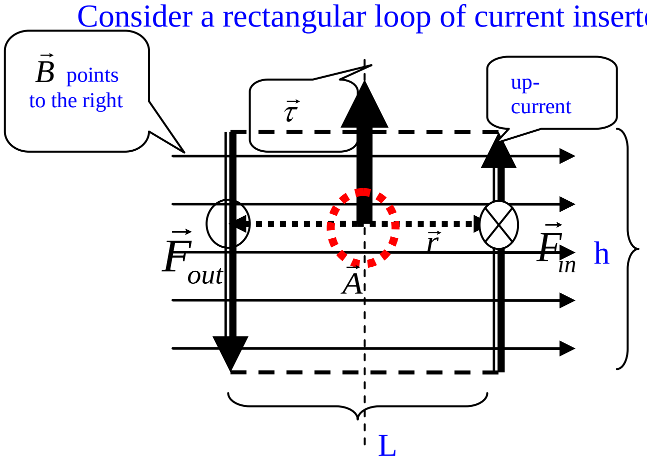
(29.20) $$\boxed{\vec{F}_B = I\vec{L}' \times \vec{B}}$$

$\vec{L}'$ is the distance vector connecting the initial point of the wire with its final point of the portion inserted into the uniform magnetic field

The net magnetic force acting on any ***closed current loop*** in a **uniform** magnetic field is 0.

29.5 Torque on a current loop in a magnetic field.

Even though the total force on a current loop in a magnetic field is zero, the torque is not. Consider a rectangular loop of current inserted in a uniform magnetic field. **The magnetic field points to the right in the adjacent picture**, and the rectangular loop of current lies in the same plane as the magnetic field. The currents in opposite sides of the rectangle move in opposite directions. Therefore the forces **on the horizontal parts of the loop are 0 because the currents are parallel (or anti-parallel) to the magnetic field**. All forces are perpendicular to the plane. Now, if the force on the left vertical line points outside of the plane, the force on the right vertical line points into the plane, thus creating a torque around the vertical axis. Both torques create a counter clockwise rotation and add up to a resultant torque which points upward along the axis of rotation.



This is a typical case of a torque created by a so-called vector couple: Two equal and opposite forces are applied at the two end-points of a bar with length 2r which is capable to rotate around its center point.

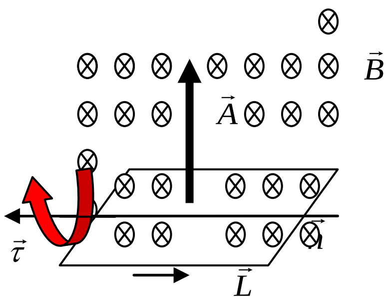
(29.21) $$\boxed{\vec{\tau} = 2\vec{r} \times \vec{F}}$$

The vector $\vec{r}$ points from the center of rotation to the endpoint at which the force $\vec{F}$ is applied.

If all relevant quantities are at right angles to each other we get the magnitude for the torque as:

(29.22) $\tau = \underbrace{2r}_{L}\,\underbrace{F}_{IhB} = \underbrace{Lh}_{A=area} \cdot I \cdot B = AIB = area \cdot current \cdot magnetic\ \ field$

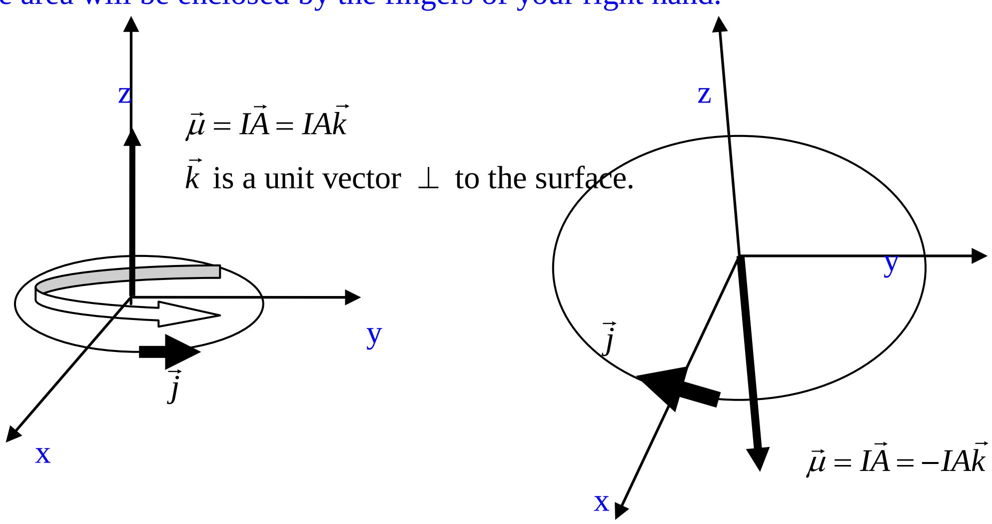
We define the area of the loop with the current I as a surface-vector $\vec{A}$ perpendicular to the surface area and with the magnitude of the area. We define the mathematics by always circulating around an area along its boundary **with the fingers curling in the direction of the positive current, and the thumb pointing in the direction of the surface vector.**



We define the **magnetic dipole moment** $\vec{\mu}$ as the vector of magitude I (current) times A and in the direction of the surface vector $\vec{A}$.

(29.23) $\boxed{\vec{\mu} = I\vec{A} = \text{magnetic dipole moment}}$

The direction of the magnetic moment can be obtained in the easiest way by curling the fingers of the right hand along the loop with the direction of the current. (We assume a current of positive charges; for negative charges the direction is reversed.) **Your thumb will then point in the direction of the magnetic moment**. In this way, the area will be enclosed by the fingers of your right hand.



When the magnetic dipole moment is placed into a magnetic field, a torque is being created which re-orients the dipole. The magnetic field will line up the magnetic moment with itself, thus obtaining a position with minimum potential energy for the dipole-magnetic field system. Using the notation of the dipole moment the torque can be conveniently written as:

(29.24) $$\boxed{\vec{\tau}_B = \vec{\mu} \times \vec{B} = I\vec{A} \times \vec{B}}$$

The torque of the magnetic moment of a positive charge current is parallel to the axis of rotation.

The torque is 0 when the surface is perpendicular to the magnetic field, i.e. when the surface vector and therefore the magnetic dipole are parallel to $\vec{B}$.

This is in analogy to what happens when we put an electric dipole into an electric field:

(29.25) $$\vec{\tau}_E = \vec{p}\times\vec{E}\ ; \quad \vec{p} = qd\vec{u}$$

where $\vec{u}$ is a unit vector pointing from -q to +q.

If a coil consists of N turns of wire, the magnetic moment of the loop is N times the magnetic moment of a single loop:

(29.25) $$\vec{\mu} = NI\vec{A}$$

Just like the potential energy of an electric dipole immersed in an electric field was given by $U_e = -\vec{p}\cdot\vec{E}$ we define the potential energy of the magnetic loop as the work necessary for an outside agent to rotate the loop in the magnetic field:

(29.26) $$\int_{\theta_1}^{\theta_2} dW = \int_{\theta_1}^{\theta_2} \tau d\theta = \mu B\int_{\theta_1}^{\theta_2} \sin\theta d\theta = -\mu B\left(\cos\theta_2 - \cos\theta_1\right)$$

We choose our reference angle for the potential energy at 90°, when the surface vector of the dipole is perpendicular to the magnetic field, i.e. when the loop lies in the plane of the magnetic field.

Thus, the potential energy of a magnetic dipole immersed in a magnetic field is given by:

(29.27) $$\boxed{U_B = -\vec{\mu}\cdot\vec{B}}$$

When the dipole is parallel to the magnetic field the potential energy is smallest(-μB), when the dipole is anti parallel it is highest (+μB). (This is the same as with the energy of the electric dipole.)

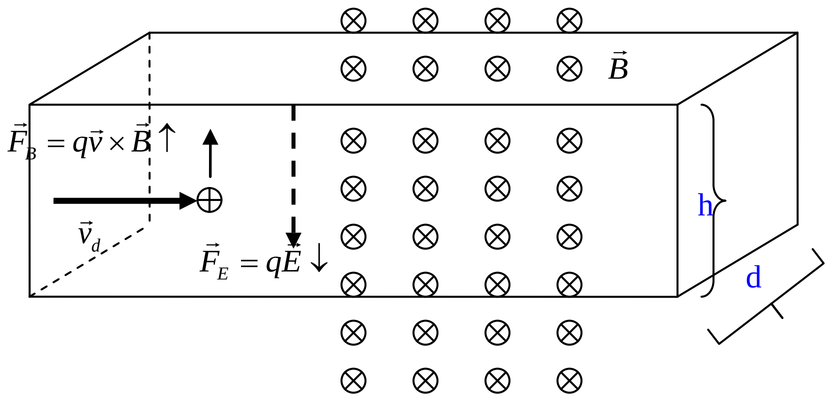
29.6 The Hall Effect :

If a current carrying conductor is placed into a magnetic field, charges are deflected to one side of the conductor, thus **creating an electric field** across the conductor, because of an excess accumulation of one kind of charges on one side of the conductor. The ensuing electric force on the deflected (by B) charges opposes the magnetic force. The accumulation stops when the electric force due to the accumulated charge surplus equals the magnetic force responsible for the deflection.

(29.28) $$q\mathrm{v}_d B = qE$$

By measuring the voltage difference across the conducting slap V_H and the current I we can determine the charge density in the conducting material. Let us assume it has the shape of a rectangular slap with cross-section dimensions of base d = 1.00 mm and height h=2.00cm, we get:

(29.29) $$\Delta V_H = Eh = v_d Bh$$



Thus we can measure the drift velocity v_d. By measuring the Hall potential and the current I we have a way to determine the moving charge density nq. We need to express the drift velocity by the current:
The drift velocity is related to the current density and the current itself.

(29.30) $$I = \vec{j} \cdot \vec{A} = n_V q \mathrm{v_d} \underbrace{hd}_{A}$$

(29.31) $$\mathrm{v_d} = \frac{I}{n_V q h d}; \Delta V_H = Eh = \mathrm{v}_d B h = \frac{I}{n_V q \cancel{h} d} B \cancel{h} = \frac{IB}{n_V q d}$$

(29.32) $$\Delta V_H = \frac{IB}{n_V q d} = R_H \frac{IB}{d}$$
$$n_V = \frac{IB}{qd}$$

By measuring the Hall voltage, the current, the magnetic field we can determine the density of the conducting charges and their sign. R_H=1/nq is called the **Hall coefficient, which is the inverse of the charge density ρ_q.**

Example: A copper strip 2 cm (h) wide and d=1mm thick is inserted into a magnetic field B perpendicular to the strip width. B=200T and I=200A. Calculate the Hall voltage.
We find: j=I/A =I/dh = 1000 A/cm^2. n=$\rho_m \cdot N_A/M_{mol}$=8.95·6E23/64=8.4E22/cm^3. This number coincides well with the concept of 1 free electron per atom in copper as the conducting electron.
E_H=0.149V/m; V_H= 2.98mV.

Such measurements allow one to measure the actual charge density in any conducting or semi-conducting material, and the sign of the charge carriers. It was a big surprise when physicists found that in some some semi-conductors the moving charges were not electrons, but positively charged holes.

Addendum:

A) Derivation of the law of Biot-Savart.

Abstract: The same mathematical differential equations have the same solutions. Underlying differential equations for the gravity potential, the electric potential, and the components of the magnetic vector potential share the same differential equations. Therefore their solutions are the same also. The differential equations in question here are the Poisson equations, or Laplace equations.

(29.33)

$$div\vec{E}=\frac{\rho}{\varepsilon_0};\vec{\nabla}\times\vec{E}=0\Rightarrow\vec{E}=-\overrightarrow{grad}\cdot V;-div(gradV)=\frac{\rho}{\varepsilon_0};$$

$$div(gradV)\equiv\vec{\nabla}\cdot\vec{\nabla}V=\Delta V=-\frac{\rho}{\varepsilon_0}\text{ Poisson equation for the electric potential V.}$$

$$\text{Poisson equation:}\left\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\rangle^2 V=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)V(x,y,z)=-\frac{\rho}{\varepsilon_0}$$

$$curl(gradV)\equiv\vec{\nabla}\times\vec{\nabla}V=0$$

(29.34)

$$div\vec{B}=0;curl\vec{B}=\mu_0\vec{j}\Rightarrow\text{there is no scalar field from which }\vec{B}\text{ derives. But }\vec{B}=curl\vec{A}=\vec{\nabla}\times\vec{A}$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=-\mu_0\vec{j}\text{ (with the choice div}\vec{A}\text{=0) this leads to the similar equation for the components}$$

of the vector potential $\vec{A}$ as for the scalar potential V.

$$\vec{\nabla}\cdot\vec{\nabla}\vec{A}=\Delta\vec{A}=-\mu_0\vec{j};\vec{j}=q\vec{v}$$

We get the same results for the potential functions caused by a single charge q

(29.35)

$$V(r)=\frac{q_1}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}_1|}\Rightarrow\text{ if }q_1\,is\text{ located at }\vec{r}_1=0\;V(r)=\frac{q_1}{4\pi\varepsilon_0}\frac{1}{r}$$

In relativistic physics it turns out that V and $\vec{A}$ form a single four-dimensional vector. The electric field and the magnetic field are also components of a four dimensional quantity, called a tensor. This drives home the fact, that the physical facts related to electromagnetism are relativistic and four dimensional in their very nature.

(29.36)

$$\vec{A}(r)=\frac{\mu_0}{4\pi}\frac{q_1\vec{v}_1}{|\vec{r}-\vec{r}_1|};\;\vec{r}-\vec{r}_1=\text{radius vector from the charge }q_1\text{ at }\vec{r}_1\text{ to point }\vec{r}\text{ of the field.}$$

Experiments show that magnetic field lines have no sources, the field lines close on themselves. There are no magnetic charges from which field lines emerge. Thus, magnetic field cannot be described through a local relationship like the electric fields of electrostatics, where electric fields start in a positve charge and end in a negative charge.

(29.36) $$div\vec{E}=\frac{\rho}{\varepsilon_0};\ curl\vec{E}=0\Rightarrow\vec{E}=-\overrightarrow{grad}\cdot V$$

This means that the electric field at a location x,y,z is due to the local change of a scalar field V. Local change of a field means that it is the result of the local vector operator $\vec{\nabla}$.
Magnetic fields never start in charges, therefore :

(29.36) $$div\vec{B}=0 \text{ always}$$

Magnetic fields curl around wires with currents in them or even around single moving charges, which therefore create a current density :

(29.36) $$\boxed{curl\vec{B}=\mu_0\vec{j}}$$

$$\frac{\mu_0}{4\pi}=10^{-7}\frac{Ns^2}{C^2};\ \mu_0=1.26\cdot 10^{-6}=\text{permeability}$$

(29.36) the dimension of $\text{curl}\times\vec{\text{B}}$ is T/m, that of current density $\vec{\text{j}}$ is $\frac{\text{A}}{\text{m}^2}$

$$\left[\vec{\nabla}\times\vec{B}=\mu_0\vec{\text{j}}\right]\ \Rightarrow\ \text{the dimension of } \mu_0 \text{ is } \frac{\text{T/m}}{\text{A/m}^2}=\frac{Tm}{A}$$

We ask ourselves, just like in the case of the electrostatic field, if the magnetic field at a point can be the result of the local change of another field.

As the $curl\vec{B}$ is different from 0, the vector field $\vec{B}$ cannot derive ($\vec{\nabla}$) from a potential scalar field, and the evaluation along a closed loop is not equal to 0. (The concepts of conservative fields do not apply!!!!!) What could be the local change of a magnetic field: theoretically, there are only three choices: $\vec{B}$ can be the grad of a scalar field, which is **exluded** by the fact that

$$curl\vec{B}\neq 0$$

it **cannot be the divergence of a vector field** because that is in itself a scalar, and the magnetic field is vector, so it could only be the curl of another vector field, which turns out that that is the case :

(29.36) $$\vec{B}=curl\vec{A}\equiv\vec{\nabla}\times\vec{A}$$

$\vec{A}$ is called the **vector potential**. Thus, the magnetic field is the curl of a **vector potential**. It must satisfy the fact that $div\vec{B}\equiv 0$, which is the case always because

$$div\vec{B}=0=div\left(curl\vec{A}\right)=\vec{\nabla}\left(\vec{\nabla}\times\vec{A}\right)=0 \text{ always;}$$

(29.36) compare to

$$\text{curl}\vec{E}=0=curl\left(gradV\right)\equiv\vec{\nabla}\times\vec{\nabla}V=0$$

Proof :

For any three vectors we have:

(29.36) $$\vec{\text{A}}\cdot\left(\vec{B}\times\vec{C}\right)\triangleq\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}=\vec{C}\cdot\left(\vec{\text{A}}\times\vec{B}\right)=\vec{B}\cdot\left(\vec{C}\times\vec{\text{A}}\right)$$

If any of these vectors are parallel, the mixed product is 0. (Properties of the determinant.)

We get from this that

(29.36)
$$curl\vec{B} \equiv \vec{\nabla}\times\vec{B} = \mu_0\vec{j} \text{ and } \vec{B} = curl\vec{A} \equiv \vec{\nabla}\times\vec{A}$$
$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \mu_0\vec{j}$$

We evaluate the double cross product according to the rules of vector operators :

(29.36)
$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}\left(\vec{a}\vec{c}\right) - \vec{c}\left(\vec{a}\vec{b}\right)$$
$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \mu_0\vec{j} = \vec{\nabla}\underbrace{\left(\vec{\nabla}\cdot\vec{A}\right)}_{div\vec{A}} - \underbrace{\vec{A}\left(\vec{\nabla}\cdot\vec{\nabla}\right)}_{-\Delta\vec{A}}$$

By setting the expression $div\vec{A} = 0$ we are fixing a constant of integration for the vector potential. We recognize the remaining operator as the ***Laplace operator*** which we encountered earlier in chapter 25.
The equation for the vector potential A is a ***Poisson equation*** just like the one for the electric potential:

(29.36)
$$\Delta\vec{A} = -\mu_0\vec{j} = -\mu_0\rho\vec{v} \text{ just like } \Delta V = -\frac{\rho}{\varepsilon_0}$$
$$\Delta A_x = -\mu_0 j_x \text{ and } \Delta A_y = -\mu_0 j_y \text{ and } \Delta A_z = -\mu_0 j_z$$

As the same equations have the same solutions we can immediately write down the solution for the vector potential. Reviewing the electric potential, we found the solution for a single charge q

(29.36)
$$V(r) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r} \text{ from } \Delta V = -\frac{\rho}{\varepsilon_0}$$

Each component of the vector potential obeys the same Poisson equation

(29.37)
$$\Delta A_x = -\mu_0\rho v_x; \Delta A_y = -\mu_0\rho v_y; \Delta A_z = -\mu_0\rho v_z$$

Therefore the solutions for the vector potential must be similar to those for the electric potential, the only difference lying in the constant factor:

(29.38)
$$A_x = \frac{\mu_0}{4\pi}\frac{\rho v_x}{r}; A_y = \frac{\mu_0}{4\pi}\frac{\rho v_y}{r}; A_z = \frac{\mu_0}{4\pi}\frac{\rho v_z}{r}$$

In the previous equtations we placed the charge density into the origin. If we place the charge density into a location with the radius vector $\vec{r}_1$, the distance vector from the charge to the point where we calculate the field is $\vec{r} - \vec{r}_1$ with the distance $\left|\vec{r} - \vec{r}_1\right|$. and now:(29.38)

$$\vec{A}(r) = \frac{\mu_0}{4\pi}q_1\frac{\vec{v}_1}{\left|\vec{r} - \vec{r}_1\right|} \text{ which is the vector potential at point } \vec{r}.$$

created by the single moving charge at the point (1):$dq_1 = \rho dV_1$

charge density $\frac{\rho}{\varepsilon_0} \Rightarrow$ single charge $\frac{q}{4\pi\varepsilon_0}$

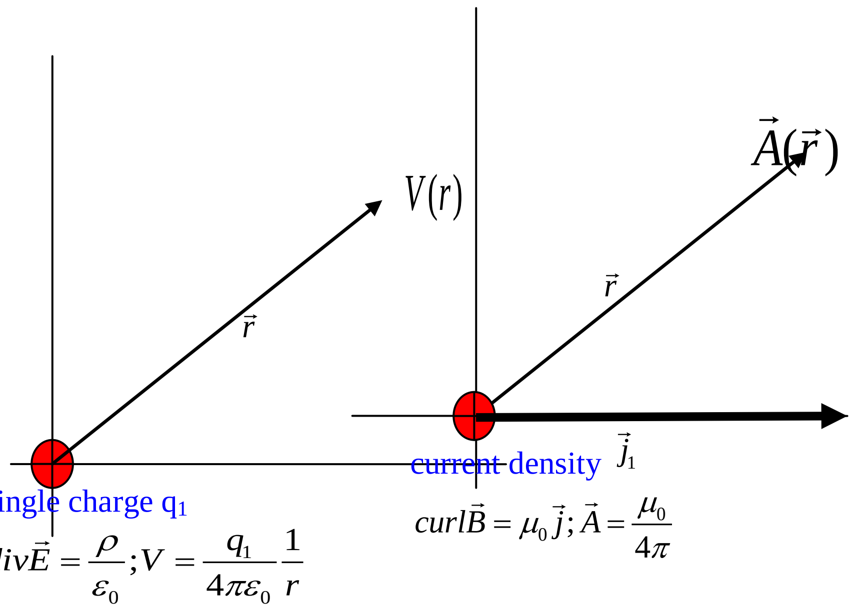
current density $\mu_0\vec{j} = \mu_0\rho\vec{v} \Rightarrow$ single charge with $\vec{v}$: $\frac{\mu_0}{4\pi}q\vec{v}$

To get the vector field $\vec{B}(r)$ we have to calculate $\vec{\nabla}\times\vec{A}$.In order calculate a derivative of these functions we need to pay attention that we calculate the derivatives at the location of the function: (x,y,z) and not at the location of the charge (x_1,y_1,z_1).
To ease up on our writing we can put the charge back into the origin and get functions which are easier to manage:

$$V(r)=\frac{k_e q_1}{r}=\frac{k_e q_1}{\sqrt{x^2+y^2+z^2}}$$

and

$$\vec{A}(r)=\frac{\mu_0}{4\pi}q_1\frac{\vec{v}_1}{r}=\frac{\mu_0}{4\pi}q_1\frac{\vec{v}_1}{\sqrt{x^2+y^2+z^2}}$$



We obtain the magnetic field created by this single moving charge at point 1, by taking the curl of A at point $\vec{r}$ according to (29.36).

$$(29.38)\ \vec{B}(r)=\vec{\nabla}\times\vec{A}\text{ with }\vec{\nabla}\times\frac{\vec{v}}{\sqrt{x^2+y^2+z^2}}\equiv\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{v_x}{\sqrt{x^2+y^2+z^2}} & \frac{v_y}{\sqrt{x^2+y^2+z^2}} & \frac{v_z}{\sqrt{x^2+y^2+z^2}}\end{vmatrix}$$

(29.38)
$$B_x = \frac{\mu_0}{4\pi} q\left(\frac{\partial}{\partial y}\frac{\mathrm{v_z}}{r} - \frac{\partial}{\partial z}\frac{\mathrm{v_y}}{r}\right);$$
$$B_y = \frac{\mu_0}{4\pi} q\left(\frac{\partial}{\partial z}\frac{\mathrm{v_x}}{r} - \frac{\partial}{\partial x}\frac{\mathrm{v_z}}{\mathrm{r}}\right)$$
$$B_z = \frac{\mu_0}{4\pi} q\left(\frac{\partial}{\partial x}\frac{\mathrm{v_y}}{r} - \frac{\partial}{\partial y}\frac{\mathrm{v_x}}{r}\right)$$

When we carry out the derivatives we use the Cartesian forms :

(29.38)
$$\frac{1}{\mathrm{r}} = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

The partial derivative of the negative square-root results in the factor -1/2 ; the derivative of the squares with respect to y and z results in the factors 2y and 2z:

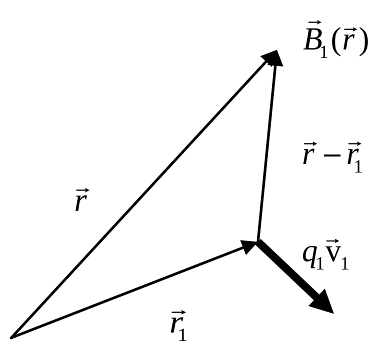
(29.39)
$$\frac{\partial}{\partial y}\left(\frac{v_z}{\sqrt{x^2+y^2+z^2}}\right) = -\frac{1}{2}\frac{2yv_z}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}$$

Therefore:

(29.39)
$$B_x = \frac{\mu_0}{4\pi} q\left(-\frac{1}{2}\right)\frac{\mathrm{v_z}2y - \mathrm{v_y}2z}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} = \frac{\mu_0}{4\pi} q\frac{\mathrm{v_y}z - \mathrm{v_z}y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}$$

We recognize that the expression in the numerator is the x-component of the cross-product $\vec{\mathrm{v}} \times \vec{\mathrm{r}}$

We get as the final result for the magnetic field $\vec{B}(\vec{r})$ created by the charge $_q$ having the velocity $\vec{\mathrm{v}}$ at location $\vec{r}$



The components in the numerator are the components of the cross-product $\vec{v} \times \vec{r}$

(29.39)
$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{q_1\vec{\mathrm{v}} \times \vec{r}}{r^3}}$$

The vector $\vec{r}$ which appears in the numerator of the equation points from the moving charge at location (1) (which in our derivation we put at the origin 0) to the magnetic field at location $\vec{r}$. If we put q_1 at location $\vec{r}_1$, the vector $\vec{r}$ must be replaced by $\vec{r}-\vec{r}_1$ and the distance r by $\left|\vec{r}-\vec{r}_1\right|$

$$\boxed{\vec{B}_1(\vec{r}) = \frac{\mu_0}{4\pi}\frac{q_1\vec{\mathrm{v}}_1 \times \vec{u}_1}{\left|\vec{r}-\vec{r}_1\right|^2}}$$

This law is also known as the law of **Biot-Savart:** If we want to calculate the magnetic field at a point $\vec{r}$, created by a small segment of current $I_1 d\vec{s}_1$ at point (1) we use see (29.13)

(29.40) $$\boxed{dQ_1 \cdot \vec{v}_1 = I_1 \cdot d\vec{s}_1}$$

We call the magnetic field $\vec{B}_1(\vec{r})$ the field created by the moving charge at the point (1)

(29.41) $$\boxed{\vec{B}_1(\vec{r}) = \frac{\mu_0}{4\pi}\frac{q_1\vec{v}_1 \times \vec{u}_1}{\left|\vec{r}-\vec{r}_1\right|^2} \Rightarrow d\vec{B}_1(\vec{r}) = \frac{\mu_0}{4\pi}\frac{I_1 d\vec{s}_1 \times \vec{u}_1}{\left|\vec{r}-\vec{r}_1\right|^2};\left|\vec{r}-\vec{r}_1\right| = \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}}$$

The unit vector points from the current element at point (1) to the point $\vec{r}$, where the magnetic field is being calculated. If we put the current element into the origin, then $\vec{r}_1 = \langle x_1, y_1, z_1 \rangle = \langle 0,0,0 \rangle$.

B) Magnetic Forces on Moving Charges, in general:

If we place another charge q_2 moving with velocity $\vec{v}_2$ into the magnetic field $\vec{B}_1(\vec{r})$ at point r, then this moving charge $q_2\vec{v}_2$ will feel the force

Force on charge (2) created by the magnetic field $\vec{B}_1$ created by moving charge q_1 :

(29.41) $$\vec{F}_{21} = q_2\vec{v}_2 \times \vec{B}_1 = q_2\vec{v}_2 \times \frac{\mu_0}{4\pi} q_1 \left(\frac{\vec{v}_1 \times \vec{u}_1}{\left|\vec{r}-\vec{r}_1\right|^2} \right)$$

Pulling the scalars out of the equation we get :

(29.41) $$\boxed{\vec{F}_{21} = q_2\vec{v}_2 \times \vec{B}_1 = \frac{\mu_0}{4\pi}\frac{q_1 q_2}{\left|\vec{r}-\vec{r}_1\right|^2}\vec{v}_2 \times (\vec{v}_1 \times \vec{u}_1)}$$

Conversely, the moving charge at location (2) creates a magnetic field $\vec{B}_2$ at location (1) and therefore exerts a force $\vec{F}_{12}$ on particle q_1.

(29.41) $$\vec{F}_{12} = \frac{\mu_0}{4\pi}\frac{q_1 q_2}{\left|\vec{r}-\vec{r}_1\right|^2}\vec{v}_1 \times (\vec{v}_2 \times \vec{u}_2)$$

The unit vectors point in opposite directions.
In general, this force is not equal and opposite to the force $\vec{F}_{21}$.

C) Magnetic forces on moving charges in parallel:

Let us see what we get when the **velocities are parallel to each other**: In that case, the unit vectors are perpendicular to the velocities. If we expand the double cross products we get:
The force on charge (1) by charge (2) is given by:

(29.41) $$\vec{F}_1 = \underbrace{\frac{\mu_0}{4\pi}\frac{q_1 q_2 \vec{\mathrm{v}}_1}{\left|\vec{r}-\vec{r}_1\right|^2}}_{k} \times \left(\vec{\mathrm{v}}_2 \times \vec{u}_2\right) = k\left[\vec{\mathrm{v}}_2 \underbrace{\left(\vec{\mathrm{v}}_1 \cdot \vec{u}_2\right)}_{0} - \vec{u}_2\left(\vec{\mathrm{v}}_1 \cdot \vec{\mathrm{v}}_2\right)\right] = -k\vec{u}_2 \mathrm{v}_1 \mathrm{v}_2$$

The force on charge (2) created by the moving charge at (1)

(29.41) $$\vec{F}_2 = k\vec{\mathrm{v}}_2 \times \left(\vec{\mathrm{v}}_1 \times \vec{u}_1\right) = k\left[\vec{\mathrm{v}}_1 \underbrace{\left(\vec{\mathrm{v}}_2 \cdot \vec{u}_2\right)}_{0} - \vec{u}_1\left(\vec{\mathrm{v}}_2 \cdot \vec{\mathrm{v}}_1\right)\right] = -k\mathrm{v}_2 \mathrm{v}_1 \vec{u}_1$$

As the unit vectors are opposite to each other we say that the two charges (of the same sign), moving parallel to each other, attract each other with the force :

(29.42) $$\boxed{F_{12} = \frac{\mu_0}{4\pi}\frac{q_1 q_2}{r^2}\mathrm{v}_1 \mathrm{v}_2}$$

This is a fundamental law of physics, as fundamental as Coulomb's law. Two parallel moving charges (of the same sign) exert an attractive force on each other which is proportional to the product of the charges and speeds, and inversely proportional to the distance between them.

Equal charges moving parallel attract each other, opposite charges moving parallel to each other repel each other.
In order to find the force exerted between current carrying wires, we need to first find the magnetic field created by the current in a wire. Then we apply equation (29.16) which we write in our new context as:

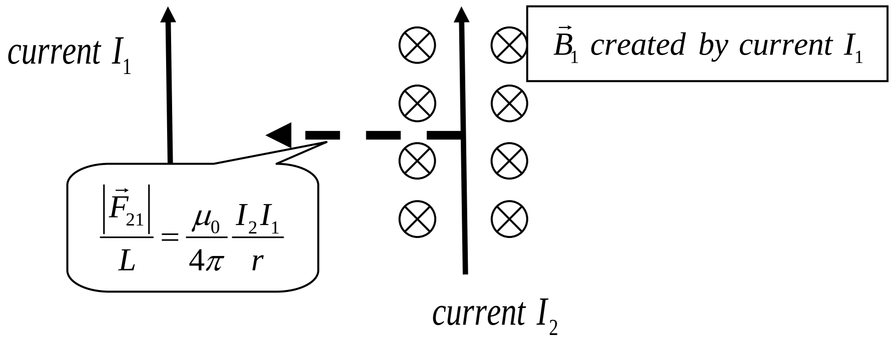
(29.43) $$d\vec{F}_2 = I_2 d\vec{s}_2 \times \vec{B}_1$$

It gives us the force on the current I_2 at location (2), created by the magnetic field $\vec{B}_1$ which is due to a current in the parallel wire (1).

We will find an easy method to calculate the magnetic field surrounding a wire with current I_1 in a later chapter. It is equal to

$$\boxed{B_1 = \frac{\mu_0 I_1}{2\pi}\frac{1}{r}}$$

and circles around the wire. The orientation of the magnetic field of current I_1 follows the right hand rule: The thumb indicates the direction of the current, and the fingers curl around it in the direction of the magnetic field. If we place a second wire parallel to the first wire, the magnetic field is perpendicular to the direction of the current. This means that the force is perpendicular to both the magnetic field and the direction of the current. This force points from one wire to the other, and is perpendicular to both parallel wires.



(29.44) $d\vec{F} = I\left(d\vec{s}\times\vec{B}\right)$ for parallel wires means that the force is perpendicular to the wires.

(29.44) $$\boxed{dF_{21} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{r} dl_2 \Rightarrow \frac{F_{2\ 1}}{L_2} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{r}}$$

Where we calculated the force on the line segment of wire (2) per unit length of the wire (2): Two currents of 1A in parallel wires, and being one m apart attract each other with the force of $2\cdot 10^{-7}$ N per meter.

D) **Calculating the magnetic field through the vector potential**:

Just as a fun exercise, let us calculate the magnetic field around a wire by using our insights into the magnetic potential $\vec{A}$

The situation is the same as with the electric potential around a wire.

For the electric field surrounding a wire with linear charge λ we found that:

(29.45) $$E(r) = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}$$

From this we calculate the electric potential at distance r from the wire by:

(29.46) $$\Delta V = -\int_{\infty}^{r} \vec{E}\cdot d\vec{s} = -\int_{\infty}^{r} \frac{\lambda}{2\pi\varepsilon_0 r}\cdot dr = -\frac{\lambda}{2\pi\varepsilon_0}\ln r\Big|_{\infty}^{r} = -\frac{\lambda}{2\pi\varepsilon_0}\ln r; V(\infty) = 0$$

Now, we can again use the fact that the electric potential and the vector potential follow the same equations, and therefore, under similar circumstances must have the same solutions :

(29.47)
$$\Delta V = -\frac{\rho}{\varepsilon_0} \text{ leads to } V(r) = \frac{-\lambda}{2\pi\varepsilon_0}\ln r$$
$$\Delta\vec{A} = -\mu_0\vec{j} = -\mu_0\rho\vec{v} \text{ must lead to } A_z = -\frac{\mu_0}{2\pi}\lambda v\ln r;\ \lambda v = \frac{dQ}{dz}\frac{dz}{dt} = I$$

We place the wire in the z-direction, and r is given by $\sqrt{x^2 + y^2}$

We have replaced ρ with $\lambda = \frac{dQ}{dz}\frac{dz}{dt} = \frac{dQ}{dt}$ =I and $\frac{1}{\varepsilon_0}$ with μ_0.

(29.48) $$A_z = -\frac{I\mu_0}{2\pi}\ln r = -\frac{I\mu_0}{2\pi}\ln\sqrt{x^2+y^2}$$

According to $\vec{B} = \vec{\nabla}\times\vec{A}$ we need only to calculate the x and y components of the magnetic field, as the component in the z-direction (the direction of the current density) is 0.

(29.49) $$B_x = \frac{\partial}{\partial y}A_z = -\frac{\mu_0 I}{2\pi}\frac{\partial}{\partial y}\left(\frac{1}{2}\ln\left(x^2+y^2\right)\right) = -\frac{\mu_0 I}{2\pi}\frac{1}{2}\frac{2y}{x^2+y^2} = -\frac{\mu_0 I}{2\pi}\frac{y}{r^2}$$

(29.50) $$B_y = -\frac{\partial}{\partial x}A_z = \frac{\mu_0 I}{2\pi}\frac{\partial}{\partial x}\left(\frac{1}{2}\ln\left(x^2+y^2\right)\right) = \frac{\mu_0 I}{2\pi}\frac{1}{2}\frac{2x}{x^2+y^2} = \frac{\mu_0 I}{2\pi}\frac{x}{r^2}$$

Which means that B circles around the wire and has the magnitude:

(29.51) $$B = \frac{\mu_0 I}{2\pi}\frac{1}{r}$$