E:\Excel files\230 lecture\ch28 DC circuits.docx;
printed:12/31/2008 12:24:00; saved: 12/31/2008 12:24:00
See website:

Chapter 28 : DC Circuits, Kirchhoffs Rules, Charging a Capacitor. (Just for reading: 28.5 Electrical Meters, 28.6 Household wiring)

Table of Contents:

28.1Electromotive Force, Internal Resistance of a Battery.

We consider circuits whose currents are typically generated through batteries. Batteries create surplus charges through chemical reactions like in the lead-acid batteries in cars. The internal resistance r of the battery can be considered as being in series with the rest of the circuit. As current I flows through the circuit it loses energy to the chemical processes. The nominal voltage or **emf (electromotive force**, for crying out loud) of a battery is the maximum voltage at which it can operate. The actual ***terminal voltage ΔV*** is less and is given by:

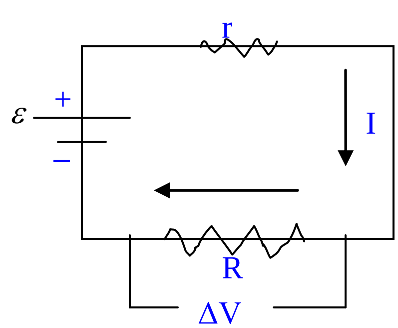
(28.1) $$\boxed{\Delta V = RI = \varepsilon - rI}$$

This can be easily seen by applying the concept of energy conservation to the whole loop: If we traverse the loop clockwise the emf is a positive value (from – to +):

(28.2) $$\underbrace{\varepsilon}_{emf} - \underbrace{rI}_{\text{voltage drop through the internal resistance}} - \underbrace{RI}_{\Delta V=\text{terminal voltage}} = 0$$

When there is no current flowing in the circuit, i.e. when the circuit is open, the terminal voltage and the emf are the same. So, when you measure the emf voltage of a battery like 1.5V (in an open circuit), this is also the terminal voltage.

However, when you install the battery into a circuit with load resistance R the situation is different. We assume a positive current to move from the positive terminal in a clockwise fashion to the negative terminal.



We define the emf as follows:

The emf is the amount of electric energy (per Coulomb of positive charge) delivered by the battery as the charge passes through the battery from the low potential terminal to the high potential terminal. Its unit is energy per Coulomb, i.e. Volt.

If a steady, time-independent current carries one Coulomb of charge around a circuit with resistance R (the load resistance), the energy lost in the circuit is equal to the energy produced in the battery (or other electric energy source).

We apply energy conservation ($\frac{Work}{q} = \frac{energy}{q} = \Delta V$) as we go around the loop with one resistor R, and one emf, and we include the internal resistance r of the battery to get:

(28.3) $$\boxed{\begin{array}{l} \varepsilon - rI - RI = 0; R \text{ is the load resistance} \\ I = \dfrac{\varepsilon}{R+r}; r = \dfrac{\varepsilon - \Delta V}{I} \end{array}}$$

You can measure ΔV in the closed circuit and calculate I. $\frac{\Delta V}{R} = I$

Caution: The internal resistance of an actual battery increases greatly as the normal alkaline battery ages. The internal resistance of an actual alkaline battery also varies with a current.

To get the power supplied by the battery to the whole circuit we multiply by the current in the circuit I:

(28.4) $$\boxed{\text{Power supplied by battery}=I\varepsilon = rI^2 + RI^2}$$

Thus, the total power delivered by the battery goes into the running of the battery, namely the energy for chemical reactions, and the running (heating) of the load R. The resistance of the load R represents any instrument, tool, etc. that is powered by the electric current. The power used by the load R is RI.

The **energy stored in the battery** is the product between the maximum charge Q (of the fresh battery) it can deliver often given in Ah (amperes times hours) and the emf.

(28.5) $$\boxed{\begin{array}{c} Q:\text{maximum charge in Ah. 1Ah=3600Coulombs} \\ E_{stored} = Q\cdot\varepsilon \end{array}}$$

Typical units of energy on small batteries are given in mAh (milli-amp-hours.)

Example: A 11Ω resistor is connected across a battery of emf 6.00 V and internal resistance 1.00 Ω. Find a) the current I in the loop b) the terminal voltage c) d) the power delivered to the external resistor and e) the power delivered to the battery's internal resistor f) If the battery is rated at 150 Ah how much energy does the fresh battery store?

from (28.3) $I = \dfrac{\varepsilon}{R+r} = \dfrac{6}{11+1} = 0.5A$ the current I in the loop

from (28.1) $\Delta V = \varepsilon - rI = 6V - 1\Omega \cdot 0.5A = 5.50V$ the terminal voltage

double check that this is the same current obtained by $\dfrac{\Delta V}{R} = \dfrac{5.50V}{11\Omega} = 0.5A$

from (28.4) $Power = I\varepsilon = rI^2 + RI^2 = 0.5A\cdot 6V = 3.00W$ the total power supplied by the chemical reactions within the battery

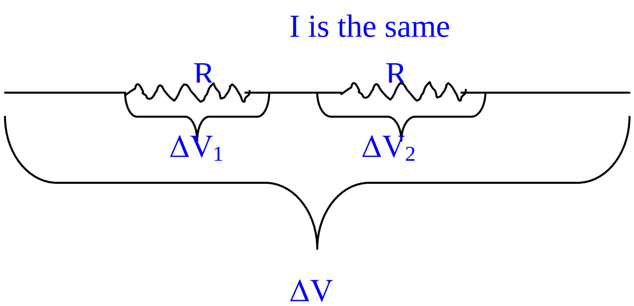
$Power = RI^2 = 11\Omega\cdot(0.5A)^2 = 2.75W$ power used by the load

$Power = rI^2 = 1\Omega\cdot(0.5A)^2 = 0.25W$ the power delivered to the battery's internal resistor

$E = Q\cdot\varepsilon = \left(150A\cdot h\cdot\dfrac{3600s}{h}\right)\cdot 6.00V = 3.24MJ$ energy stored by the fresh battery

28.2 Resistors in series and parallel.

It is easy to see that resitors which are arranged in series in a circuit can be replaced by the **equivilant resistor** which is the sum of all individual resistors. The wire does not have any branches and therefore acts like conduit: we see the same number of charges per unit time passing through any cross-section of the wire, i.e. the current is the same as long as there are no junctions.



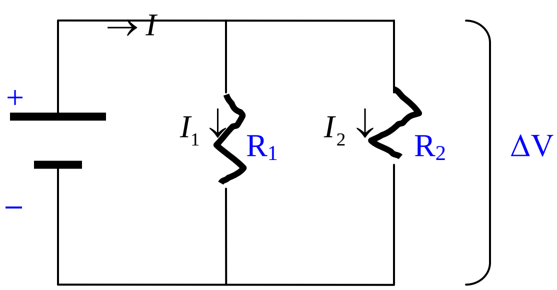
$$\Delta V = \Delta V_1 + \Delta V_2 = R_1 I + R_2 I = \left(R_1 + R_2\right) I$$

(28.6)

$$\boxed{\Delta V = \sum_k \Delta V_k = \sum_k R_k I = R_{eq} I = \left(\sum_k R_k\right) I}$$

The total voltage drop from one end to the other is equal to the sum of the voltage drops.

For a circuit in which resistors are arranged in ***parallel*** we use conservation of energy and conservation of charge to arrive at our formula for the equivalent resistance:



Conservation of charge or current at a junction point. $I = I_1 + I_2$ The potential difference is the same across both resistors.

The potential difference across two or any nmber of resistors in parallel are the same. Therefore we can write:

$$I = I_1 + I_2 = \sum_k I_k = \sum_k \frac{\Delta V}{R_k} = \frac{\Delta V}{R_{eq}}$$

(28.7)

$$\frac{1}{R_{eq}} = \sum_k \frac{1}{R_k}$$

For two resistors in parallel:

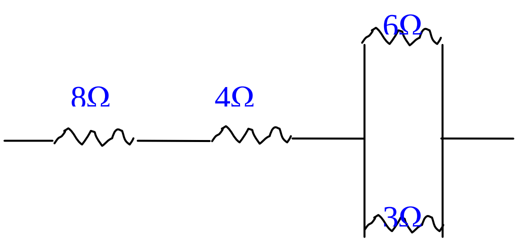
(28.8) $$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

For resistors any number of resistors in parallel we find the equivalent resistor through :

(28.9) $$\boxed{\frac{1}{R_{eq}} = \sum_k \frac{1}{R_k}}$$

Note that the equivalent resistance in a loop with all resistors in parallel, is smaller than the smallest resistor.

Examples:

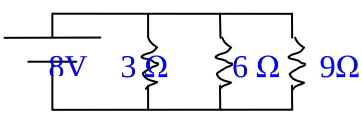


$$\frac{1}{6} + \frac{1}{3} = \frac{3}{6} \Rightarrow 2\Omega$$

$$8 + 4 + 2 = 14$$

The equivalent resistance is 14 Ohms.

In another example we have three resistors connected in parallel, with 3, 6, and 9 Ohms, respectively. They are supplied by an 18 Volt battery. Find the current in each resistor and the power delivered to them.



The equivalent resistance in this circuit can be obtained by: 1/3+1/6+1/9=0.611, the inverse of which is the equivalent resistance: 1.636 Ohms. The total current delivered by the battery is therefore: 18/1.636 =11A.
The current running through the 3 Ohm resistor is 18/3 =6A, through the 6 Ohm resistor 18/6=3A, and through the 9Ohm resistor 2A, for a total of 11A.
The power delivered to each resistor is the product between its current and its voltage, we get 108, 54, and 36 W respectively. **The smallest resistor receives the most power**. The current moves through the path of least resistance.

28.3 Kirchhoff's Rules:

For analyzing more complicated circuits it is convenient to formalize the results above in what are known as ***Kirchhoff's rules***:

- Draw a circuit diagram.
- Assign a current with direction to every branch of the circuit.
- When you traverse the various elements you must stick to the assigned direction of the current.
- Traversing (traversing means you following a rotational direction, which sometimes goes with and sometimes against the assigned direction of the currents.) an emf from – to + results in a positive value +ΔV.
- Traversing a resistor in the direction of the assigned current results in a negative value –RI, traversing it against the assigned direction results in +RI.
- **Complete all loops in either clockwise or counter-clockwise fashion**. The loop rule says that the sum of all voltage drops in a closed loop is 0. It's a good idea to start and end a loop on both sides of the emf.

Don't mix the orientations.

- The junction rule says that the algebraic cum of the currents at any junction point is 0.

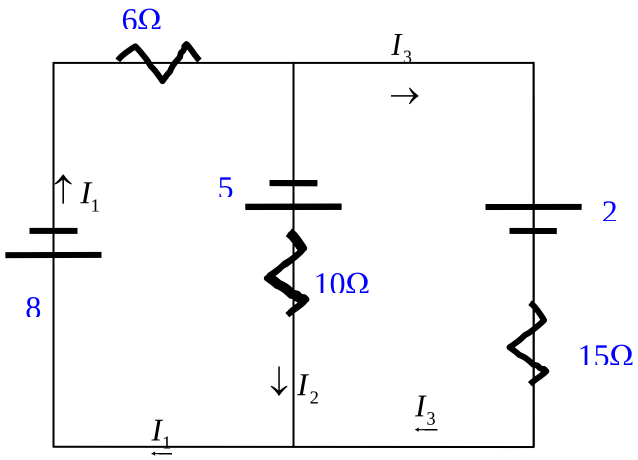
(28.10) $$\boxed{\sum_{\substack{k=every\\ junction}} I_k = 0;\ \text{charge conservation, current in = current out}}$$

- Apply the loop rule to all complete loops required. The sum of all algebraic values of the potential differences encountered in completing a loop is equal to 0.

(28.11) $$\boxed{\sum_k \Delta V_k + \sum_l \varepsilon_l = 0 \text{ for every loop; conservation of energy for a complete loop.}}$$

In this sum, the emf of a source is counted as positive if the current flows through the source in the forward direction (from – to +) and negative in the backward direction (from + to -). The voltage drops through resistors are negative if counted in the direction of the current and positive if the resistor is traversed in the opposite direction of the current direction assigned to this segment of loop..

- Solve the system of independent linear equations for the currents.



Clockwise:

$-8V-6I_1+5V-10I_2=0$

$-5V-2V-15I_3+10I_2=0$

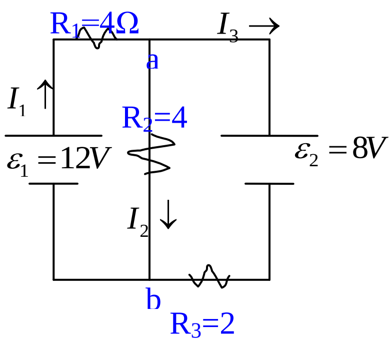
$I_1-I_2-I_3=0$

$$\begin{pmatrix} -6 & -10 & 0 \\ 0 & 10 & -15 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix}$$

$A \cdot X = B \Rightarrow X = A^{-1}B$

$I_1 = -.48A; I_2 = -.010A; I_3 = -.473A$

Another example:



Assign currents through each resistor (current subscript = resistor subscript) in a clockwise direction. For the loop to the left we get:

$12-4I_1-4I_2=0$

For the loop to the right we get:

$-8-2I_3+4I_2=0$

At a junction a we get:

$I_1-I_2-I_3=0$

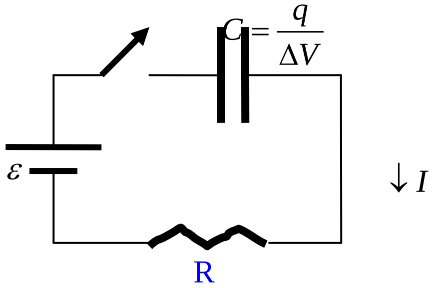
The result is $I_1 = 1.25A; I_2 = 1.75A; I_3 = -0.5A$

28.4 Capacitors in DC circuits, RC circuits.

When we install a capacitor in an open circuit consisting of an emf, a resistor, and a switch, there is no current running until we close the switch. At that instant the capacacitor will start to be charged, and consequently a current will flow until the capacitor is charged to the max. The positive side of the capacitor coincides with the positive side of the emf. When we traverse the circuit loop from the negative terminal of the emf to the positive terminal, we get the following equation:

28.4a Charging a capacitor:

In the moment we close the switch a current flows through the circuit and the capacitor gets charged with dq until it has reached its maximum possible charge Q=C$\cdot\varepsilon$.



(28.12) $$\varepsilon - RI - \frac{q}{C} = 0 \text{ or } \varepsilon - R\frac{dq}{dt} - \frac{q}{C} = 0$$

q =q(t) is the varying charge, the maximum charge is $Q = C\varepsilon$. The current I and the charge q are functions of time.

This equation is simple to integrate by separation of the variables, q to the left and t to the right. In the time t the charge increases from 0 to q :

(28.13) $$\varepsilon - R\frac{dq}{dt} - \frac{q}{C} = 0 \Rightarrow \varepsilon - \frac{q}{C} = R\frac{dq}{dt} \Rightarrow C\varepsilon - q = RC\frac{dq}{dt} \Rightarrow$$
$$\frac{dq}{C\varepsilon - q} = \frac{dt}{RC} \Rightarrow \frac{dq}{q - \varepsilon C} = -\frac{dt}{RC}$$

(28.14) $$\int_0^q \frac{dq}{q - C\varepsilon} = \frac{-1}{RC}\int_0^t dt \Rightarrow \ln\left(q - C\varepsilon\right)\Big|_0^q = \frac{-t}{RC} + a$$
$$q - C\varepsilon = be^{\frac{-t}{RC}} \Rightarrow q = be^{\frac{-t}{RC}} + C\varepsilon; q(0) = 0 = b + C\varepsilon \Rightarrow b = -C\varepsilon$$

(28.15) $$q\left(t\right) = C\varepsilon\left(1 - e^{\frac{-t}{RC}}\right) = Q\left(1 - e^{\frac{-t}{RC}}\right)$$

Thus, the formula for the increasing charge on a capacitor being charged for the time t, is :

(28.16) $$\boxed{q = C\varepsilon\left(1-e^{\frac{-t}{RC}}\right) = Q\left(1-e^{\frac{-t}{RC}}\right)}$$

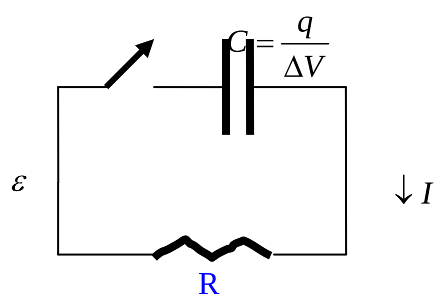
The factor RC is called the time-constant: $\tau = RC$
We can see that the halftime of the circuit is equal to

(28.17) $$t_{\frac{1}{2}} = \tau \ln 2$$

We get the expression for the charging current by simply taking the derivative with respect to time of the charge q(t) :

(28.18) $$\boxed{I = \frac{dq}{dt} = \frac{C\varepsilon}{RC}e^{\frac{-t}{RC}} = \frac{\varepsilon}{R}e^{\frac{-t}{RC}}}$$

28.4b Discharging a capacitor:
When we want to discharge the capacitor through the resistor, we need to eliminate the emf in (28.12). In the moment we close the switch the capacitor starts to discharge through the resistor.



The circuit equation is simply:

(28.19) $$-I(t)R - \frac{q(t)}{C} = 0 \text{ or } \frac{dq}{dt} = \frac{-q}{RC} \Rightarrow q(t) = Qe^{\frac{-t}{RC}}$$

We get the discharging current again through differentiation :

(28.20) $$I = \frac{dq}{dt} = -\frac{Q}{RC}e^{\frac{-t}{RC}} = -I_0 e^{\frac{-t}{RC}}$$

To measure the current with an Ammeter we insert the ammeter into the branch in which we want to measure the current. This is called putting the Ammeter **in series**. The Ammeter should have 0 resistance. (In Series.)

To measure the voltage across a resistor, between any two points in a circuit, we attach the leads on the outside, i.e. we **measure in parallel**. The resistor should have infinite (10MΩ) resistance.