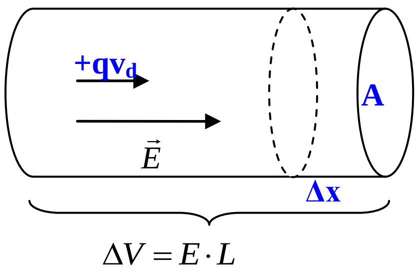
Homework: See website.

Table of Contents:

Ch. 27 Electric Current and Resistance.

27.1 Electric Current I and current density j,



We consider a conducting wire with length L and cross-section A in a constant electric field E, directed parallel to the cylinder's axis.
Free positive charges flow to the right.

(27.1)
$$I \equiv \frac{dQ}{dt} = \text{flow rate of charge} = \text{current}$$
$$[\mathrm{I}] = \frac{Coulomb}{time} \equiv Ampere; 1A$$

We assign the positive direction of current to the direction of flow of **positive** charges, i.e. the actual direction of flow (drift velocity v_d) of electrons on a wire is opposite to the conventional direction of current. The electric field is directed from + positive polarity to – negative polarity.
(Pitfalls: distinguish carefully between v for velocity, V for volume and ΔV for potential difference; also there are two densities: the conventional mass density ρ or ρ_m; charge density ρ_q; and we use another symbol ρ later, the resistivity ρ_Ω.)

27.1a Review of incompressible liquid flow as example for flux of an electric field:
We discussed this previously at the introduction of the concept of flux through a surface which led us to the Gaussian law for electrostatics: $div\vec{E} = \frac{\rho}{\varepsilon_0}$ (chapter 24, chapter 20).

Let us review again what we learned about the rate of **water flow** out of a cylindrical pipe:

The amount of liquid mass flowing out of the pipe per second is given by $\frac{dM}{dt} = \rho_m A\mathrm{v}$

If we had a number N of identical particles, each with mass m, flowing through the pipe, we could write: $\rho_m = n_V m$, with n_V (number-density) $= \frac{N}{V}$ and define the mass current density $\vec{\mathrm{j}} = \mathrm{n}_V \cdot \mathrm{m} \cdot \vec{\mathrm{v}}$. The mass current would then be equal to the scalar product of current density and cross-sectional surface: $\frac{dM}{dt} = \vec{\mathrm{j}} \cdot \vec{A}$

(27.2)
$$\text{Current}: \; I = \frac{dM}{dt}$$

For a steady state flow of an incompressible liquid, i.e. with constant mass density, we found that the amount of liquid coming in is equal to the amount of liquid coming out. This was referred to as the continuity equation. We can see that the continuity equation is a special case of the law that says that the divergence of the current density is equal to 0, which means that the total flux of the current density through a closed surface is 0. This result presumes that there are no liquid sources or sinks for liquid inside of the volume.

(27.3) $$div\cdot\vec{j}=0\Rightarrow\oiint_{\substack{Gaussian\\surface}}\vec{j}\cdot d\vec{A}=0\Rightarrow j_1A_1=j_2A_2$$

27.1b Electric current and electric current density:

In complete analogy with this we get for the **electric current I** of positive electric charges q through a conducting wire:

(27.4) $$\boxed{\begin{aligned}&I=\frac{dQ}{dt}=\rho_q\frac{d}{dt}(Adx)=\rho_qAv=\underbrace{n_Vq\vec{v}}_{\vec{j}}\cdot\vec{A}\\&[I]=\frac{Coulomb}{s}=\frac{C}{s}=ampere=A\\&[j]=\frac{A}{m^2}\end{aligned}}$$

$$I=\vec{j}\cdot\vec{A}=\underbrace{n_Vq}_{\rho_q}\underbrace{A\frac{dx}{dt}}_{\frac{dV}{dt}}=\frac{\text{number of charges}}{\text{volume}}\text{ elementary charge}\cdot\text{cross-section}=\rho_qA\mathrm{v}_\mathrm{d}$$

(27.5) $I=n_Vqv_dA$ where v_d is the drift velocity of the charges due to the electric field $\vec{\mathrm{E}}$.

q is the value of the elementary charge $=1.60\cdot10^{-19}Coulomb$

Conversely, we can define the magnitude of the **current density** "j" as the current divided by the cross-section A:

(27.6) $$\text{current density}=\mathrm{j}=\frac{\mathrm{I}}{\mathrm{A}}\text{ or in vector form}$$
$$\mathrm{I}=\iint_A\vec{\mathrm{j}}\cdot d\vec{A}=\vec{\mathrm{j}}\cdot\vec{\mathrm{A}}=\text{flux of the charge-density vector through the surface A}$$

(27.7) $$\boxed{\vec{j}=n_Vq\vec{\mathrm{v}}_\mathrm{d}=\rho_q\vec{\mathrm{v}}_\mathrm{d}}$$

27.1c Density ρ_q of conduction electrons:

The **density** of conducting electrons in copper wire is obtained by assuming that there is one conducting electron per copper atom. The molar mass of copper is 63.5 g/mol. The mass density of copper is 8.95 g/cm^3.

(27.8) $$\frac{8.95\frac{g}{cm^3}}{63.5\frac{g}{mol}}=0.141\frac{mol}{cm^3}=0.141N_A\cdot10^6\frac{atoms}{m^3}=n_V$$

(27.9) number density of conducting electrons in Cu: $n_V=8.5\cdot10^{28}$ electrons per m^3

conducting charge density in copper $\rho_q=n_Vq=8.5\cdot10^{28}\cdot1.9\cdot10^{-19}=1.62\cdot10^{10}\frac{C}{m^3}$

Note: Get **clear on the various distinctions between current I, and current density j, as well as charge density, mass density, mass current, and mass current density**.

27.2 Electric Resistance in Conducting Material, conductivity, resistivity:
The German physicist Georg Simon Ohm (1789-1854) established that for **conventional metals** there is a direct proportionality between the current density and the applied electric field (Note that this is not always the case). Materials in which the **current density is directly proportional to the applied electric field** are called Ohmic:

(27.10)
$$\boxed{\begin{aligned}&\vec{j} = \sigma\vec{E} = conductivity \cdot \vec{E};\ \text{Ohm's law}\\&\vec{E} = \frac{1}{\sigma}\vec{j} = \rho_\Omega \vec{j};\ \text{with } \rho_\Omega = \frac{1}{\sigma} = \text{resistivity}\\&[\rho_\Omega] = \frac{[E]}{[j]} = \frac{V}{m}\frac{m^2}{A} = \frac{V}{A}m = Ohm - meter = \Omega m\end{aligned}}$$

Pay attention to whether the Greek symbol (sigma) σ refers to the conductivity $\frac{1}{\rho_\Omega}$ or to the surface density of charges $\frac{Q}{A}$.

Resistivity ρ_Ω is the inverse of conductivity σ, thus **<u>Ohm's law</u>** can also be written as:

(27.11)
$$\boxed{\vec{E} = \rho_\Omega \vec{j}}$$

From the concept of **conductivity** we get to the concept of **electric resistance** R in a wire by applying Ohm's law above to a segment of wire with length L to which a voltage ΔV is applied. Across the length of a wire with length L we have $\Delta V = E \cdot L$ (ΔV depends on the initial and final point only, not on the path or shape of the wire)

We write for the magnitudes:

(27.12)
$$\begin{aligned}&j = \sigma E\\&\Delta V = E \cdot L = \frac{j}{\sigma}L = \frac{I}{A}\cdot\frac{L}{\sigma} = \frac{1}{\sigma}\frac{L}{A}I = \underbrace{\rho\frac{L}{A}}_{R}I\end{aligned}$$

Solving this for ΔV

(27.13)
$$\boxed{\Delta V = \frac{1}{\sigma}\frac{L}{A}I = \rho_\Omega\frac{L}{A}I = R \cdot I = \text{resistance} \cdot \text{current}}$$

This is the most common form of Ohm's law, which states that the voltage difference measured across a wire is equal to the electrical resistance of the wire times the current. The unit of resistance is called Ohm for which we use the Greek capital letter for R

(27.14)
$$\boxed{\begin{array}{l} \Delta V = RI = \text{resistance}\bullet\text{current} \\ [R] = \dfrac{Volts}{amperes} = \dfrac{V}{A} = \Omega \end{array}}$$

(27.15)
$$\boxed{\begin{array}{l} \vec{E} = \sigma\vec{\mathrm{j}};\ \vec{\mathrm{j}} = nq\vec{v};\ I = \iint\limits_A \vec{\mathrm{j}}\cdot d\vec{A};\ \Delta V = RI \\ \rho_\Omega = \dfrac{1}{\sigma} \equiv \text{resistivity in Ohm-meters} \\ \mathrm{R}=\rho_\Omega \dfrac{L}{A} \text{ in Ohms or } \Omega \end{array}}$$

The **resistance R** of a wire increases with the length and decreases with the cross-section.
The resistivity of copper is 1.7E-8 Ω m. Both, resistivity and conductivity are material constants, which can be looked up in tables. They are usually given at a temperature of 20°C. They both vary with temperature: **resistivity and resistance increase with temperature in most metals**.

27.2a Calculating R for varying cross-sections:
Sometimes we have to consider a flow of current through a non-constant cross-section, like, for example in a cylinder or sphere. I such cases we need to consider the current flow through infinitesimal sections and integrate:

Example: Find the radial resistance between the walls of a coaxial wire of length 1m with inner radius 2mm and outer radius 4mm. The walls consist of plastic material with resistivity 1.0E13 Ω m.

(27.16)
$$dR = \rho_\Omega \frac{dr}{2\pi r L} \Rightarrow R = \frac{\rho_\Omega}{2\pi L}\int_a^b \frac{dr}{r} = \frac{\rho_\Omega}{2\pi L}\ln\frac{b}{a} = \frac{\rho_\Omega}{2\pi L}\ln 2 = 1.1\cdot 10^{12}\,\Omega$$

Find the radial resistance between the interior and exterior walls of a spherical shell:

(27.17)
$$dR = \rho_\Omega \frac{dr}{\pi r^2} \Rightarrow R = \frac{\rho_\Omega}{\pi}\int_a^b \frac{dr}{r^2} = -\frac{\rho_\Omega}{\pi}\frac{1}{r} = \frac{\rho_\Omega}{\pi}\left(\frac{1}{a}-\frac{1}{b}\right) = 8.0\cdot 10^{14}\,\Omega$$

27.3 Simple classical model for electrical conduction:

Conducting electrons drift through a metal due to the **electric force** $q\vec{E}$. This means they are being accelerated by the electric field. They do not move in a straight line but bounce off of positively charged areas. Due to this effect electrons reach a final velocity (like terminal velocity), or average drift velocity v_d which we can estimate in the following way:
Between two scattering points the electrons are being accelerated due to the applied exterior electric field. $F = qE = m\ddot{x} = ma;\ a = \dfrac{qE}{m}$

$\vec{v}_d = \vec{v}_0 + \vec{a}\tau$; where τ is the average time interval between individual scatterings, the corresponding length would be the mean free path length for conducting electrons.

$v_d = \frac{l}{\tau}$

average $\vec{v}_0 = 0$

Note: Do not mix this free path length up with the free path length of random thermodynamic motion. They are similar but not the same.

$$v_d = at = \frac{qE}{m}\tau$$

We insert this drift velocity into our formula for the current density j; (27.7):

(27.18) $$j = \sigma E = n_V q v_d = \frac{n_V q^2 E}{m}\tau$$

(27.19) $$\boxed{\sigma = \frac{n_V q^2}{m}\tau \text{ and } \rho = \frac{1}{\sigma} = \frac{m}{n_V q^2 \tau}}$$

According to this classical model the conductivity and resistivity do not depend on the current, (or voltage or electric field.) We can calculate the time between collisions for copper which has a resistivity of 1.7E-8 Ω m and **n=8.49E28** electrons/m^3 (See (27.9)).

(27.20) $$\tau = \rho \frac{m}{n_V q^2} = \frac{1}{\sigma}\frac{m}{n_V q^2}$$

It turns out to be about 2.5E-14 s.

The drift velocity, on the other hand does depend on the current.

(27.21) $$I = jA = n_V q v_d A \Rightarrow v_d = \frac{I}{n_V q A}$$

This means that the **drift velocity** v_d and the distance between scatterings increases with increasing currents, whereas the time between scatterings remains the same.
For a typical 10A current in a 12 gauge copper wire the cross sectional area is 3.309 mm^2. See table (27.27). (Note that the smaller the gauge number, the larger the diameter; household wire is typically 12 and 14 gauge; the diameter for 12 gauge wire is 2.053 mm at 20 degree C, 14 gauge is 1.628 mm, 10 gauge is 2.588 mm). This results in a drift velocity of 0.222 mm/s.

27.4 Resistance and Temperature.

Over a limited temperature range **the relative change of resistivity** is proportional to temperature:

$$\rho = \rho_0\left[1+\alpha\left(T-T_0\right)\right]; \text{ and } T_0 \text{ are usually taken at } 20°Celsius$$

(27.22) $$\frac{\Delta\rho}{\rho_0} = \alpha\Delta T; [\alpha] = \frac{1}{Celsius°}; \alpha = \frac{1}{\rho}\frac{d\rho}{dT} \Rightarrow \rho = \rho_0 e^{\alpha(T-T_0)}$$

α is called the temperature coefficient of resistivity. (For Pt: $\alpha=3.92\cdot10^{-3}/C°$)
As the resistance R in a circuit is proportional to the resistivity, we get the same formula for R.

(27.23) $$\boxed{\frac{\Delta R}{R_0}=\alpha\Delta T;\alpha=\frac{1}{R}\frac{dR}{dT}\Rightarrow R=R_0e^{\alpha(T-T_0)}}$$

Evaluating this between R_0 and R leads to the expression for the temperature dependence of resistance in a common metal.

(27.24) $$\alpha=\frac{1}{R}\frac{dR}{dT}\Rightarrow\int_{R_0}^{R}\frac{dR}{R}=\alpha\int_{T_0}^{T}dT\Rightarrow\ln\frac{R}{R_0}=\alpha\left(T-T_0\right)$$
$$R=R_0e^{\alpha(T-T_0)};e^x\approx1+x\Rightarrow R=R_0(1+\alpha\left(T-T_0\right))$$

(27.25) $$R=R_0\left[1+\alpha\left(T-T_0\right)\right];\left[\alpha\right]=\frac{1}{Celsius°}$$

27.4a Gauge measures for electric wire,

The most accurate thermometers are based on the changing resistivity and resistance of a particular kind of wire (often platinum). We used such a thermometer in the mechanical equivalent of heat experiment, where we determined the relationship between R and T to be of a logarithmic nature, in its best approximation.
Here are some constants:

(27.26) $$\begin{aligned}&Cu:\rho_\Omega=1.7\cdot10^{-8}\Omega m;\alpha=3.9\cdot10^{-3}/C°\\&Pt:\rho_\Omega=11\cdot10^{-8}\Omega m;\alpha=3.92\cdot10^{-3}/C°\\&Nichrome:\rho_\Omega=1.50\cdot10^{-6}\Omega m;\alpha=0.4\cdot10^{-3}/C°\end{aligned}$$

Note that Nichrome wire has about 100 times the resistivity of copper but only 1/10 of the temperature coefficient. This leads to the common use of Nichrome wire in heating elements like toasters, irons, and electric heaters. The thickness of electric wire is rated by gauge: the higher the gauge number, the smaller the diameter of the wire.

(27.27) $$\begin{aligned}&\#10\ d=0.1019"=2.58826mm\\&\#12\ d=0.0808"=2.05232mm\\&\#16\ d=0.0508"=1.29032mm\\&\#20\ d=0.032"=0.8128mm\\&\#22\ d=0.0254"=0.64516mm\end{aligned}$$

Find the resistance per meter of #16 Nichrome wire:

(27.28) $$\frac{R}{L}=\frac{\rho_\Omega}{A}=\frac{1.50\cdot10^{-6}}{\pi d^2/4}=1.15\frac{\Omega}{m}$$

If a voltage difference of 110 Volt is maintained across 1 m of such wire, find the current in the wire:

(27.29) $$I=\frac{\Delta V}{R}=\frac{110V}{1.15\Omega}=95.7A$$

The resistance of a similar copper wire would be about 100 times smaller. A potential difference of only about 1V would result in the same current in the wire. As we will see in the next section, power delivered to an electric circuit is equal to $P=RI^2$, and in a wire this power is turned into heat, which, in turn will increase the resistance.

Some materials, called superconductors, lose suddenly all resistance at a low critical temperature T_c. The phenomenon was first discovered in mercury, which has a critical temperature of 4.15ºK. Some of the highest critical temperatures can be found in ceramics with T_c up to 134ºK.

27.6 Electrical Power.

When current travels through a resistor, heat is being generated and electric potential energy lost, (Ohm loss.) The power supply provides the same potential difference across the boundaries of a resisting material; ΔV is therefore constant.

(27.30)
$$\boxed{\begin{array}{l}\frac{dU}{dt}=\frac{d}{dt}(Q\Delta V)=\frac{dQ}{dt}\Delta V=I\Delta V=I\cdot RI=I^2R\\ \text{As } I=\frac{\Delta V}{R}\text{ this is also equal to }\frac{dU}{dt}=Power=\frac{(\Delta V)^2}{R}\end{array}}$$

Power delivered to a resistive wire or appliance:

(27.31)
$$\boxed{Power=\frac{(\Delta V)^2}{R}=I^2R}$$

Example:

A household appliance is rated according to the maximum power it consumes. As appliances typically run at 110 Volts, we can calculate the maximum current "drawn" by the appliance. A 100 Watt light bulb, for example, draws a current of 100 Watts/110Volts =0.91 Amps. Every house is subdivided into areas of 10 and 15 Amps, each protected by a circuit breaker. Whenever too many appliances are running at the same time and the total current drawn exceeds 10 or 15 Amps, the circuit breaker gets activated and stops the flow of electricity. If an unprotected circuit gets overloaded, the wiring is in danger of overheating and eventually **causing a fire. (Even though households are using ac current, the same concepts of current and power appl**y.)

Problem: A coil of Nichrome wire is 25.0 m long. The wire has a diameter of 0.400mm at 20ºC. If it carries a current of 0.500A, calculate the electric field in the wire and the power delivered to it. If the temperature is increased to 340ºC and the voltage across the wire remains constant, what is the power delivered at the higher temperature.

$\alpha=0.4\cdot10^{-3}/C°;\rho_\Omega=1.50\cdot10^{-6}\Omega m$?

(27.32)
$$\Delta V=El=RI;R=\rho_\Omega\frac{l}{A};$$
$$A=\frac{\pi d^2}{4}=1.26\cdot10^{-7}m^2;R=299\Omega;E=\frac{\Delta V}{l}=\frac{RI}{l}=\frac{299\cdot0.5}{25}=5.97V/m$$

(27.33)
$$P=RI^2=I\Delta V=74.6W\text{ with }\Delta V=RI=150V$$

If the temperature is increased the resistance R increases. To maintain the same voltage drop, the current must decrease.

(27.34) $R' = R_0 e^{\alpha(T-T_0)} = 299\Omega e^{0.0004 \cdot 320} = 299\Omega \cdot 1.1366 = 340\Omega$ (not 337 which you get with 1+x)

(27.35) $$I' = \frac{150V}{R'} = 0.4414A$$

The power delivered is therefore:

(27.36) $$P' = R'(I')^2 = 66.2W$$