Homework: See website.

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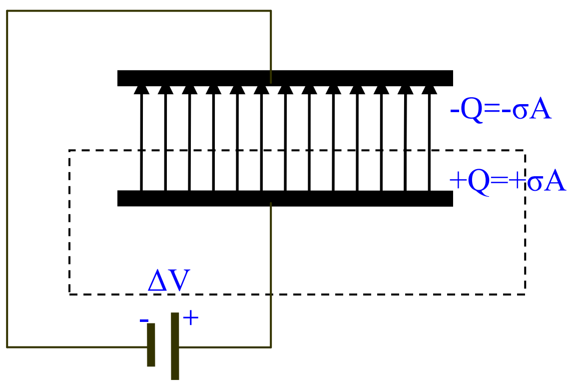
Ch.26 Definition of Capacitance.
The capacitance is the factor of proportionality between voltage ΔV and charge Q. The higher the capacitance the more charge can be put on a device called the capacitor.

$$Q = C \cdot \Delta V \quad (26.1)$$

$$C \equiv \frac{Q}{\Delta V} \text{ in the units of } \frac{\text{Coulomb}}{\text{Volt}} \equiv \text{Farad}$$

We have already calculated the electric field between two parallel plane **conductors** charged with equal and opposite charges. As we are dealing with a conductor, the electrons put on one plate will push the electrons on the opposite plate away (they flow into the battery.) As a result we have a constant electric field between the two plates of the capacitor.

26.1a Parallel Plate Capacitor.
We can easily find the electric field by using the now familiar Gauss law. Choose the Gaussian surface to be the dashed rectangle:



$$\iiint_{volume} div\vec{E}dV = \oiint_{\substack{Gauss\\surface}} \vec{E}d\vec{A} = \iiint_{volume} \frac{\rho}{\varepsilon_0} dV$$

$$= EA = \frac{\sigma A}{\varepsilon_0}; \text{ hence E=} \frac{\sigma}{\varepsilon_0}$$

$$\Delta V = -\int_A^B \vec{E} \bullet d\vec{s} = -Ed$$

$$C = \frac{Q}{\Delta V} = \frac{\sigma A}{d \, \sigma / \varepsilon_0} = \frac{\varepsilon_0 A}{d} \quad (26.1)$$

We see that the capacitance of a **parallel plate capacitor** is proportional to the area divided by the distance between the plates.

The surfaces with opposite charges are called plates. The whole arrangement is called a capacitor, in our case a **parallel plate capacitor**. It is used to store electric charge and energy. A new quantity is being used in this context, which is the ratio between the **positive charge and the positive voltage difference**. This ratio (always positive) is called capacitance, and is abbreviated with the symbol *C* (in italic lettering, in contrast to Coulomb which is a normal non-italic C) The capacitance depends only on the geometry of the device. ΔV is proportional to the charge, and therefore drops out of the ratio Q/ΔV. The capacitance depends only on the electric constant and the geometry of the capacitor.

(26.1)
$$C \equiv \frac{Q}{\Delta V} \text{ in the units of } \frac{\text{Coulomb}}{\text{Volt}} \equiv \text{Farad}$$

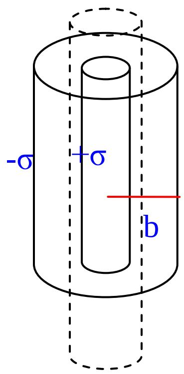
$$1Farad \equiv 1F = \frac{1C}{1V};$$

(26.2)
$$1micro - Farad \equiv 1\mu F = 10^{-6} F;$$
$$1pico - Farad \equiv 1pF = 10^{-9} F$$

Cylindrical and spherical capacitors:

Let us next calculate the capacitance of a **cylindrical capacitor** and a **spherical capacitor**. In both cases we have an outer and inner charge Q.

In the case of the cylinder we have an inner cylinder with radius “a” and an outer cylinder with radius “b”. The thickness of each cylindrical surface is negligible. Let us calculate the electric field using Gauss law, and then obtain the potential difference through integration. The electric field is perpendicular to the cylindrical surfaces. It is not constant. Therefore, the calculation of ΔV leads to an integral.



(26.3)
$$\oiint_{\substack{Gaussian\\surface}} \vec{E}d\vec{A} = \frac{Q}{\varepsilon_0} = \frac{2\pi a L\sigma}{\varepsilon_0} = \underbrace{E2\pi rL}_{EA}; \quad E = \frac{a\sigma}{\varepsilon_0}\frac{1}{r}$$

$$C = \frac{Q}{\Delta V} = \frac{2\pi a L\sigma}{\Delta V}$$

So we need ΔV:

(26.4)
$$|\Delta V| = \left| -\int_a^b \frac{a\sigma}{\varepsilon_0}\frac{1}{r}dr \right| = \frac{a\sigma}{\varepsilon_0}\ln\frac{b}{a} \Rightarrow C = \frac{\varepsilon_0 2\pi a L\sigma}{a\sigma \ln\frac{b}{a}} = \frac{2\pi\varepsilon_0 L}{\ln\frac{b}{a}} = \frac{L}{2k_e \ln\frac{b}{a}}$$

26.2b Capacitance of a Cylindrical Capacitor:

(26.5)
$$\boxed{C = \frac{L}{2k_e \ln\frac{b}{a}} = \frac{2\pi\varepsilon_0 L}{\ln\frac{b}{a}}}$$

We could have also calculated the charge by using Q=λL.

Note: the relationship between line-charge λ, surface charge σ, and volume charge ρ can be obtained by realizing that:

(26.6)
$$Q = \lambda l = \sigma A = \rho V$$

A capacitor like this can be found in a **coaxial cable**, which consists of two concentric cylindrical conductors separated by an insulator.

26.2c Capacitance of two conducting spheres: (The total charge on one surface is Q)
For the electric field between two conducting spherical shells we get

$$E = k_e \frac{Q}{r^2}$$

(26.7) $$\Delta V = -k_e Q \int_a^b \frac{dr}{r^2} = k_e Q\left(\frac{1}{b} - \frac{1}{a}\right) = k_e Q\left(\frac{a-b}{ab}\right)$$

Taking the absolute value, we re-arrange to $k_e Q\left(\frac{b-a}{ab}\right) = |\Delta V|$

The charge cancels out and we get C=Q/ΔV

(26.8) $$\boxed{C = \frac{ab}{k_e \;\; b-a}}$$

If the radius of the outer sphere goes to infinity we get the capacitance of a simple sphere. (a becomes negligible in the denominator, then b cancels out):

(26.9) $$C = \lim_{b\to\infty} \frac{ab}{k_e \;\; b-a} = \frac{a}{k_e} = 4\pi\varepsilon_0 a$$

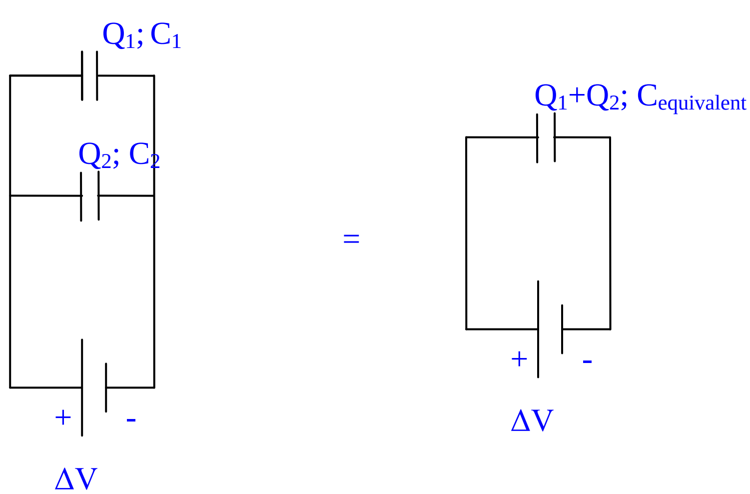
(26.10) $$\boxed{C = \frac{a}{k_e} = 4\pi\varepsilon_0 a}$$

26.3 Combinations of Capacitors in Parallel and Series.

Two or more capacitors in parallel: The left sides of the capacitors are all connected to the **same conducting wire, which means they are at the same voltage**, the same is true for the right side of the capacitors. This means that all capacitors in parallel have the same voltage difference but their total charge is the charge on all capacitors combined.

(26.11) $$C_i = \frac{Q_i}{\Delta V}$$

The equivalent capacitor is one single capacitor which has the same capacitance as all the capacitors combined in parallel. **As the positive plates are connected by a conducting wire the total charge on the two plates is equal to the sum of the charges**. $Q = Q_1 + Q_2$
The exterior voltage is provided by a battery.



(26.12)

All capacitors are at the same voltage: $\Delta V = \Delta V_1 = \Delta V_2 = ...\Delta V_i$

The total charge is the sum of the charges of the individual capacitors: $Q=\sum_i Q_i = \sum_i C_i \Delta V$

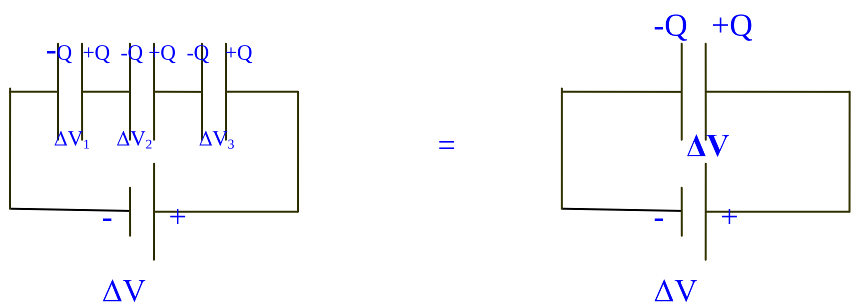
The equivalent capacitance is defined through: $Q=C_{eq}\Delta V$

$$Q=C_{eq}\Delta V = \sum_i Q_i = \sum_i C_i \Delta V \Rightarrow C_{eq} = \sum_i C_i$$

The equivalent capacitance of a number of capacitors in parallel is the sum of all capacitances.

(26.13)
$$\boxed{C_{eq} = \sum_i C_i}$$

When we put a number of capacitors **in series** inside of a circuit, we can see immediately that now the **charge on every capacitor plate is the same** and the total voltage difference across all the capacitors is the sum of the individual voltage differences of all capacitors.



The total voltage difference is the sum of the individual voltage differences. Each voltage difference has the same charge.

(26.14)
$$\Delta V = \frac{Q}{C_{eq}} = \sum_i \Delta V_i = \sum_i \frac{Q_i}{C_i} = Q\sum_i \frac{1}{C_i}$$

(26.15)
$$\boxed{\frac{1}{C_{eq}} = \sum_i \frac{1}{C_i}}$$

The inverse equivalent capacitance of a number of capacitors in series is the sum of all inverse capacitances.

26.4 Energy Stored in a Charged Capacitor.

Recall our definitions for work and capacitance in the case of a voltage across a parallel plate capacitor: Note: we use the following convention: Q (capitalized) is the charge on the capacitor, q (small cap) is the charge we move against the potential difference ΔV, which in a capacitor is equal to Q/C. For a single charge q to be added to the charge Q, already present on a capacitor with ΔV between the plates, the battery, charging the capacitor must do work:

(26.16)
$$W = q\Delta V; C = \frac{Q}{\Delta V}; \Delta V = \frac{Q}{C} \Rightarrow W = q\frac{Q}{C}$$

Let us change the single charge into a small (infinitesimal) amount of charge dq.
In order to add this small amount of charge dq to the positive plate of a capacitor with the variable charge q on it we have to do infinitesimal work against the electric field (against the potential difference ΔV) inside of the capacitor. We increase the charge q from 0 to Q.

(26.17)
$$dW = \underbrace{\Delta V}_{\substack{\text{across the}\\ \text{capacitor}}} \cdot dq = \frac{q}{C}dq$$
$$W = \int_0^Q \frac{q}{C}dq = \frac{1}{2}\frac{Q^2}{C}$$

This work done to the capacitor appears as electric potential energy U of the capacitor:

(26.18)
$$\boxed{U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C\left(\Delta V\right)^2;\ \text{with } Q=C\Delta V}$$

(Note: the work done against the electric field is equal to the potential energy of the electric field.)

For a parallel plate capacitor we found that $C = \frac{\varepsilon_0 A}{d}$. The potential difference is simply $\Delta V = Ed$. Combining this we get for the potential energy of a parallel plate capacitor:

(26.19)
$$U = \frac{1}{2}\frac{\varepsilon_0 A}{d}\left(Ed\right)^2 = \frac{1}{2}\varepsilon_0 E^2 \underbrace{Ad}_{\text{volume}}$$

Thus, we can define the **electric energy density u_E** inside the space between the plates of capacitors as:

(26.20)
$$\boxed{u_E = \frac{1}{2}\varepsilon_0 E^2;\quad u_E = \frac{U}{\text{volume containing E}}}$$

This result is correct for the electric energy density in any electric field.

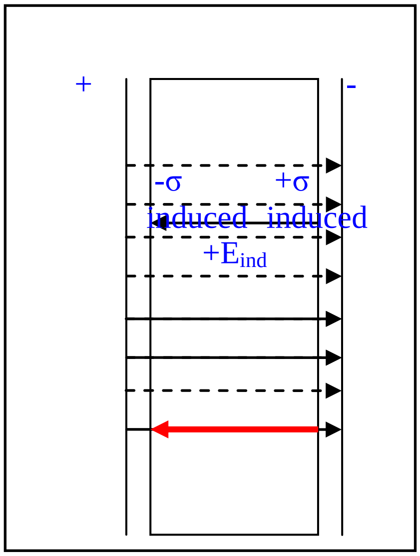
<u>26.5 Capacitors With Dielectrics</u>.
By filling the space between the plates of a capacitor with non-conducting, so-called **<u>dielectric</u>** material we can increase the capacitance C. (Wood, glass, paper) As C is defined as Q/ΔV the effect of the dielectric is to **reduce the potential difference and the electric field** strength and increase the capacitance. $\Delta V = \frac{Q}{\kappa C}$. The charge remains unaffected, as we can see from the picture below.

If we call the capacitance without the dielectric C_0 then the capacitance with the dielectric is given by:

(26.21)
$$C = \kappa C_0;\ \kappa(kapa) = \text{dielectric constant} > 1$$

The capacitance for a parallel plate capacitor with a dielectric is:

$$C = \kappa C_0 = \kappa\frac{Q}{\Delta V_0} = \kappa\frac{\sigma A}{E_0 d} = \kappa\frac{\sigma A}{\sigma/\varepsilon_0\, d} = \kappa\frac{\varepsilon_0 A}{d}$$



The **dielectric strength** in Volts/m of a dielectric gives the maximum voltage per unit length (of the thickness of the dielectric material), beyond which the dielectric material becomes conductive. The original electric field from + to -, induces an opposite electric field E_{ind} across the dielectric material, which reduces the original electric field E_0 .

Example: We use a parallel plate capacitor with dimensions 2cm by 3cm, separated by 1.0 mm layer of paper with a dielectric constant of 3.7. Its dielectric strength is $16\cdot10^6$ V/m.

The capacitance is

(26.22) $$C=\kappa\frac{\varepsilon_0 A}{d}=\frac{3.7\varepsilon_0 6\cdot10^{-4}m^2}{10^{-3}m}=20\,pF;\ \text{with } \kappa=3.7$$

Find the maximum charge that can be put on such a capacitor. $Q_{\max}=C\Delta V_{\max}$ As the thickness of the dielectric is1mm, the maximum voltage which we can put on this capacitor is given by

(26.23) $$\text{Dielectric strength}=16\cdot10^6\frac{V}{m}=16\cdot10^3\frac{V}{mm}\Rightarrow \Delta V=16,000V$$

equal to $16\cdot10^3$ Volts.

(26.24) $$Q_{\max}=20\,pF\cdot16\cdot10^3 Volts=320\cdot10^{-9}Coulombs=0.32\mu Coulombs$$

The introduction of a dielectric material into a capacitor **decreases the original field strength**, both V and E, but leaves the charge unaffected:

(26.25) $$C=\kappa C_0=\frac{Q}{\Delta V}=\kappa\frac{Q}{\Delta V_0}=\frac{Q}{Ed}=\kappa\frac{Q}{E_0 d}$$

(26.26) $$\Delta V=\frac{\Delta V_0}{\kappa}\ and\ E=\frac{E_0}{\kappa}$$

.

Calculate the energy inside of a capacitor with the dielectric substance inserted:
We saw in (26.18) that the potential energy of a capacitor is equal to ½ Q^2/C

(26.27) $$U=\frac{1}{2}\frac{Q^2}{C}=\frac{1}{2}C\ \ \Delta V\ ^2=\frac{1}{2}\kappa C_0\left(\frac{\Delta V_0}{\kappa}\right)^2=\frac{U_0}{\kappa}$$

This means that the total energy of a capacitor goes down by $1/\kappa$

This was to be expected because $\Delta V=\dfrac{\Delta V_0}{\kappa}$

What happens when we insert some molecules into the inside of a capacitor with an electric field? Provided that the molecules have an asymmetrical charge distribution, they form a natural dipole. The dipoles experience a torque due to the electric field and get lined up with the electric field lines. In order to understand better what happens in such situations we need to study the **dipole inserted into an electric field**. (See ch25 electric potential.docx page 13.

26.6 Electric Dipole in an Electric Field.

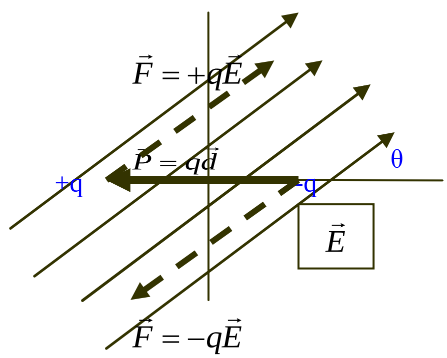
Earlier, we defined the concept of the electric dipole moment: (the word moment is a historical use of a length. For example: the magnitude of the moment of inertia is *mass times distance squared*, the ***dipole moment is charge times distance.***). Think of a dipole as a stick of length d=2a with a positive charge on one end and a negative charge on the other end.

(26.28)

$$\vec{p} = 2aq\vec{u}_r = q\vec{d}$$

$\vec{u}_r$ is the unit vector pointing from the negative to the positive charge q. and "a" is the half-distance between the two charges. Note that this direction is opposite to that of an electric field which is directed from + to -.

If we insert a dipole into an electric field, the electric field will exert a force $+q\vec{E}$ on the positive charge and a force $-q\vec{E}$ on the negative charge. As the charges are bound to the dipole structure (molecule), the electric field will produce a torque around the center of the dipole.



You can see that the torque attempts to rotate the dipole until it is lined up with the electric field, which is the most stable equilibrium position, corresponding to the minimum amount of potential energy of the dipole-electric field system.

(26.29)

$$\vec{\tau} = \vec{r} \times \vec{F}; \vec{r} \text{ points from the center of rotation to the charge.}$$

$$\vec{\tau}_1 = \vec{a} \times q\vec{E}; \vec{\tau}_2 = -\vec{a} \times \ -q \ \ \vec{E}$$

Both torques have the same direction (into the plane) and produce a resultant torque, which tries to line up the dipole with the electric field, such that the vector orientation of the dipole is parallel to the exterior electric field.

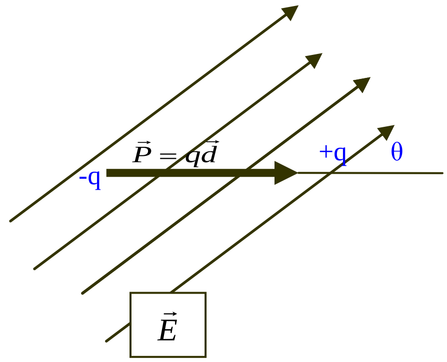
(26.30)

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 = 2\vec{a} \times q\vec{E} = \vec{p} \times \vec{E}$$

If an outside agent rotates the dipole inside of the field it will have do work, which is equal to the potential energy change of the dipole in the field (Note the changed orientation of the dipole in the figure below, as compared to the previous figure!):

(26.31) W done on the dipole by the outside agent $= \Delta U = U_2 - U_1 = \int_{\theta_1}^{\theta_2} \tau d\theta = \int_{\theta_1}^{\theta_2} pE\sin\theta d\theta =$

$$= -pE\cos\theta\Big|_{\theta_1}^{\theta_2} = -pE\left(\cos\theta_2 - \cos\theta_1\right)$$

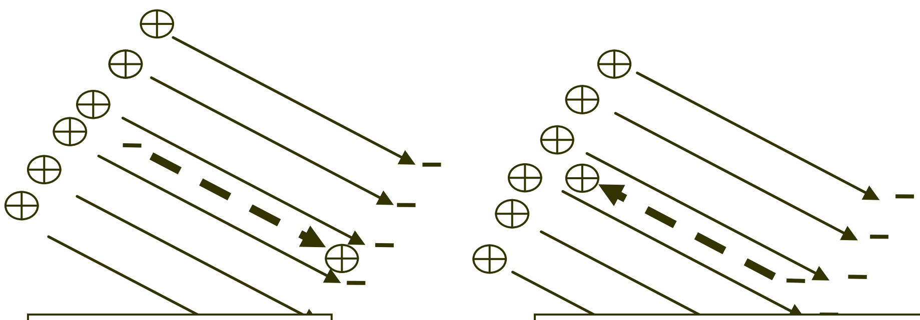


It is convenient if we choose our **reference** angle in such a way that the potential energy equals 0 **when the direction of the dipole is perpendicular to the electric field**, which means we choose our reference angle as π/2. That choice allows us to define the potential energy of the dipole inside of an electric field as a scalar product, and ensures that the position of minimum energy corresponds to that, in which the direction of the dipole and the electric field are parallel.

(26.32)
$$\boxed{U_{dipole} = -\vec{p}\bullet\vec{E}}$$

This potential energy has a minimum when the dipole and the electric field point in the same direction (θ=0º) and is equal to –p·E. It has a maximum when the vectors are anti-parallel, and U=+pE. It takes a maximum "effort" of energy for an outside agent to twist the dipole from its parallel position, to its antiparallel position. The maximum change of the dipole potential energy is 2pE, corresponding to the rotation of the dipole from an aligned position (-pE) to an anti parallel position +pE. $\Delta U = U_f - U_i = pE - (-pE) = 2pE$

The minimum value of the energy of the "dipole-electric field system" corresponds to -pE, (the two vectors $\vec{p}$ and $\vec{E}$ are parallel, which is shown in the next figure on the left. As the potential energy of the dipole-E system is $U = -\vec{p}\cdot\vec{E}$ parallel vectors means minimum potential energy. In a manner of speaking: "The positive charge of the dipole is held in place by the negative end of the electric field vector, and the negative charge of the dipole is held in place by the positive end of the electric field vector." The minimum energy position corresponds to the "natural" alignment between dipole and electric field.



$\vec{p} \parallel \vec{E} \Leftrightarrow U = -\vec{p} \cdot \vec{E}$
minimum U

$\vec{p}$ anti-parallel to $\vec{E} \Leftrightarrow U = +\vec{p} \cdot \vec{E}$
maximum U

When the outside agent releases the dipole, the electric field rotates the dipole back into the aligned position with minimum potential energy -pE.
Compare this to mgy which is the maximum potential energy of a mass lifted up from 0 to a height y. When you release the mass it will be forced back by the downward gravitational force towards its position with minimum potential energy 0.
The relationship between a conservative force and its potential energy can be generalized to the relationship between a torque and its potential energy. It is this generalized concept which dictates the definition of the dipole's direction from – to +.

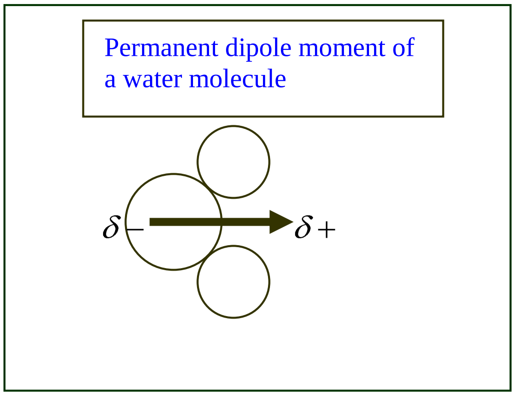
(26.33)
$$U = -\vec{p} \cdot \vec{E} = -pE\cos\theta$$
$$\vec{\tau} = \vec{p} \times \vec{E}; \ \ \tau = pE\sin\theta = -\frac{\partial U}{\partial \theta'}; \theta' = \frac{\pi}{2} - \theta$$

We defined U=0 with reference to a position in which the dipole and the electric field are perpendicular to each other. A rotation from this position involves the angle θ' which is complementary to the angle θ between $\vec{p}$ *and* $\vec{E}$ as defined in the torque.

Dipoles occur naturally in molecules, when the negative charges are not symmetrically distributed around a net central positive charge. Such asymmetric molecules have permanent dipole moments due to an excess of electrons on one end of the molecule, and a corresponding deficit on the other. This means that the molecule has a negative charge on one end (surplus electrons) and a positive charge on the other (deficit of electrons).

The typical example occurs in a water molecule, in which the two hydrogen atoms are arranged around the oxygen atom at a 105 degree angle, thus producing what is called a polarized molecule with a permanent dipole moment of 6.3×10^{-30} Coulomb-meter. The electrons tend to concentrate on the oxygen atom.

Using this example, let us calculate how much energy is required to rotate all molecules in one mole of water from an aligned position into a position perpendicular to the electric field of magnitude 250kV/m.



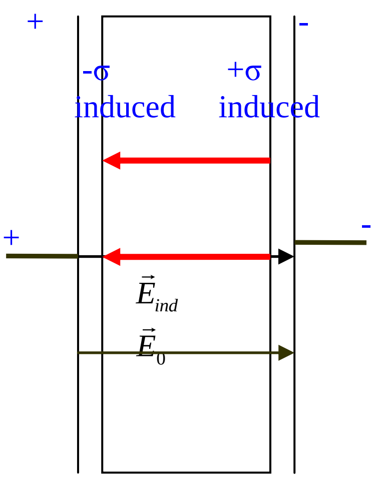
for one molecule: $W = \Delta U = -(pE\cos 90° - pE\cos 0°) = U(90°) - U(0°) = +pE = 1.6\cdot 10^{-24} J$

for N molecules: $W = 1.6\cdot 10^{-24} J \cdot 6.0\cdot 10^{23} = 0.96 J$

Doing work on the dipole increases the energy of the dipole-electric field system.
We have to do the maximum work if we want to turn all the dipole moments around to an anti-parallel position:

(26.34) $$\Delta U = U_f - U_i = -(pE\cos 180° - pE\cos 0°) = -(-pE - pE) = 2pE$$

26.7 Details about dielectrics:
Molecules that do not have a permanent dipole moment may acquire a temporary dipole moment when placed into an electric field. The opposite electric forces on the negative and positive charges of such material can distort the molecules and produce a charge separation. Such a charge separation, which lasts only as long as the material is immersed in the electric field, is called an induced dipole moment. As we can see from the picture, **the induced charges create an induced electric field opposing the original field**. Thus, in summary:



If we introduce dielectric material into a capacitor, the dipole moments of the material are lined up and form an **induced electric field inside of the dielectric, which opposes the original field E_0.**
Inside of the dielectric we therefore have a reduced resultant field E:

(26.35) $$E = E_0 - E_{ind}$$

The induced electric field originates from the surface charges of the dielectric material: $E_{ind} = \frac{\sigma_{ind}}{\varepsilon_0}$, whereas the original field is given by $E_0 = \frac{\sigma}{\varepsilon_0}$

The electric field between the plates of the capacitor with the dielectric inserted is according to (26.26) given by $E = \frac{E_0}{\kappa}$

This is the same as the resultant field.

(26.36)
$$E_{res} = E_0 - E_{ind} \Rightarrow \frac{E_0}{\kappa} = \frac{\sigma}{\varepsilon_0} - \frac{\sigma_{ind}}{\varepsilon_0}$$
$$\frac{\sigma}{\varepsilon_0 \kappa} = \frac{\sigma}{\varepsilon_0} - \frac{\sigma_{ind}}{\varepsilon_0}$$

(26.37)
$$\frac{\sigma}{\kappa} = \sigma - \sigma_{ind} \Rightarrow \sigma_{ind} = \sigma - \frac{\sigma}{\kappa}$$

(26.38)
$$\boxed{\sigma_{ind} = \left(1 - \frac{1}{\kappa}\right)\sigma = \frac{\kappa - 1}{\kappa}\sigma}$$

(26.39)
$$a)\kappa = 3 \Rightarrow \sigma_{ind} = \frac{2}{3}\sigma$$
$$b) no\ dielectric\ \kappa = 1; \sigma_{ind} = 0$$
$c)$ the material is conductive $\Rightarrow$ E=0 $\Rightarrow E_0 = E_{ind} \Leftrightarrow \sigma_0 = \sigma_{ind}$.

The sign of the charge density on the inserted conductor is opposite that on th plates.

The induced surface charge, the induced electric field, and the induced potential difference are given by:

(26.40)
$$\boxed{\sigma_{ind} = \frac{\kappa - 1}{\kappa}\sigma;\ E_{ind} = \frac{\kappa - 1}{\kappa}E_0;\ \Delta V_{ind} = \frac{\kappa - 1}{\kappa}\Delta V_0}$$

If we insert a metallic slap of thickness "a" centrally into the space of a parallel plate capacitor with plate distance d, we create in effect two parallel plate capacitors connected in series, each having the capacitance

(26.41)
$$C_{1,2} = \frac{2\varepsilon_0 A}{d - a}$$

(26.42)
$$\boxed{C_{eq} = \frac{C_1 \cdot C_2}{C_1 + C_2} = \frac{\varepsilon_0 A}{d - a}}$$

It does not matter where we insert the conducting slap.
We see that if we make the slap very thin in comparison to the space d, the original capacitance remains unchanged.

(26.43)
$$\lim_{a \to 0} \frac{\varepsilon_0 A}{d - a} = \frac{\varepsilon_0 A}{d}$$

This leads us to the next constellation of a capacitor with a dielectric slap of thickness f•d, f being a fraction between 0 and 1, inserted into the space d between the original plates and adjacent to one plate. If we insert a thin conducting slap below the dielectric material, we, in

effect, create a series of two capacitors, one with the dielectric and thickness fd, the other without dielectric and thickness (1-f)d.
The equivalent capacitance turns out to be:

(26.44)

$$C_{eq} = \frac{C_1 \bullet C_2}{C_1 + C_2}; C_1 = \frac{\kappa \varepsilon_0 A}{fd}; C_2 = \frac{\varepsilon_0 A}{(1-f)d}$$

$$C_{eq} = \frac{\kappa}{f + \kappa(1-f)} \frac{\varepsilon_0 A}{d} = \frac{\kappa}{f + \kappa(1-f)} C_0$$