Chapter 25: Potential energy and electric potential (voltage) associated with electric charges.

Homework: See webpage

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25.1 Electrostatic Field and its Gradient:

So far we have seen that electric charges create electric fields. Electric fields can also be created in other ways, as we shall see later. Let me just mention here already that **magnetic fields do not have charges, ever**. This is expressed by the fact that the divergence of magnetic fields B, is always 0. So, they must be created through other means. To summarize what we have learnt so far:

$$div\vec{E} = \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Leftrightarrow \Phi_{E,total} = \oiint_{\partial V} \vec{E}d\vec{A} = \iiint_V \frac{\rho}{\varepsilon_0} dV = \frac{Q}{\varepsilon_0}$$

We know that if a vector field is the gradient of a scalar function, the vector field is a conservative field. We look at a review of the concepts developed in mechanics, where we studied the conservative gravitational field, the gravitational force, potential energy, and work. The same relationships apply in the case of the electrostatic field.

Apart from the constants, the gravitational field of a point mass and the electrostatic field of a negative charge are the same. This means that the electrostatic force is conservative and there must also be an electrostatic potential energy. Let us briefly review the relationship between work, force, and potential energy.

Optional: (I want to have my fun with this too!.) If the divergence of a vector field is different from 0 it is a conservative field, i.e. it derives from a scalar field. This again means that the curl of the vector field is 0. It also means that the potential field follows a so-called Laplace (or Poisson partial differential equation.)

$$div\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \vec{E} = -\overrightarrow{grad}\cdot V \Leftrightarrow curl\vec{E} = 0$$

$$and\ div\left(-\overrightarrow{grad}\cdot V\right) = \frac{\rho}{\varepsilon_0} \Leftrightarrow \vec{\nabla}\cdot\vec{\nabla}V = -\frac{\rho}{\varepsilon_0};$$

(25.1) $$\vec{\nabla}\cdot\vec{\nabla}V = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle V = \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)}_{\Delta = Laplace\ operator} V$$

The solution for a point charge of such an equation is given by:

$V(r) = \frac{q}{4\pi\varepsilon_0 r}$ and the electric field can be obtained by:

$$\vec{E} = -\overrightarrow{grad}\cdot V = \frac{q}{4\pi\varepsilon_0}\frac{\vec{u}_r}{r^2}$$

End of optional.

25.1a General Relationship between Force and potential energy:

(Much of this material is also covered in 230 ch19supp2 Vector operators.docx)

The generalized relationship between a conservative **force** $\vec{F}$ and its **potential energy U** is given in terms of the differential operator:

(25.1) $\vec{F} = -\overrightarrow{grad}U = -\left\langle \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right) \right\rangle \equiv -\left(\partial_x U, \partial_y U, \partial_z U \right) = -\vec{\nabla} U$

This is true whenever the curl of the vector field is 0. For any force to be a conservative force, its curl must be 0. This is the same as to say that its circulation is 0.

(25.2) $$\vec{F} = -\overrightarrow{grad} \cdot U \Leftrightarrow \vec{\nabla} \times \vec{F} = 0 \Leftrightarrow \oint_{\substack{closed\\loop}} \vec{F} \cdot d\vec{s} = 0$$

To simplify the writing of partial derivatives one often defines that

(25.3) $\partial_x \equiv \frac{\partial}{\partial x}; \partial_y \equiv \frac{\partial}{\partial y}; \partial_z \equiv \frac{\partial}{\partial z}$

Do not confuse this notation with subscripts for vector components, like, for example A_x which is the x-component of the vector $\vec{A}$.

Here in (25.1) we have defined $\vec{\nabla}$ as the differential vector operator in Cartesian coordinates (del-operator, an upside down Greek Δ) as:

(25.4)

$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$ This vector operator must operate on some mathematical quantity to its right; it can be a scalar function, a vector function, or another differential operator. In the case of a vector function it can operate through a dot product or a cross-product.

(25.5)

$a)\vec{\nabla} \equiv (\partial_x, \partial_y, \partial_z) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right);$

$b)\vec{\nabla} U = \overrightarrow{grad}U \text{ gradient of } U = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right)$

$c)\vec{\nabla}\vec{E} = div\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$ divergence of the vectorfield $\vec{E}$

$d)\vec{\nabla} \times \vec{B} = curl\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ B_x & B_y & B_z \end{vmatrix}$

$e)$ The del operator squared $\vec{\nabla}^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \Delta$ is equal to the scalar product between the two vector operators, and is given the name Laplace operator. Its mathematical symbol is the regular delta Δ.

for example, let $\vec{F} = \langle x^2 yz, xz, xyz \rangle$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 yz & xz & xyz \end{vmatrix} = \langle C_x, C_y, C_z \rangle ; \partial_x \equiv \frac{\partial}{\partial x} \text{partial derivative}$$

$$C_x = \partial_y F_z - \partial_z F_y = xz - x; C_y = \partial_z F_x - \partial_x F_z = x^2 y - yz;$$

$$C_z = \partial_x F_y - \partial_y F_x = z - x^2 z \text{ evidently not conservative!}$$

If the curl of a vector field $\vec{E}$ is 0, then it is the gradient of some scalar function V:

(25.6)
$$\vec{\nabla} \times \vec{E} = \vec{0} \Rightarrow \vec{E} = -grad \cdot V \equiv -\vec{\nabla} V$$
$$\vec{\nabla} \times \vec{E} = \vec{\nabla} \times \left(-\vec{\nabla} V \right) = \underbrace{-\vec{\nabla} \times \vec{\nabla}}_{\substack{\text{cross product of two parallel} \\ \text{vectors is always 0.}}} \cdot V = \vec{0}$$

(We adopt the convention to call U the potential energy and V the electric potential. The force $\vec{F}$ is related to the potential energy U, and the field $\vec{E}$ is related to the potential function V.

(25.7) $$\vec{F} = -\overrightarrow{grad} \cdot U \text{ and } \vec{E} = -\overrightarrow{grad} \cdot V \text{ because } \vec{E} = \frac{\vec{F}}{q_0} \text{ and } V = \frac{U}{q_0}$$

We know that the relationship for every potential energy function U and the associated conservative force F is

(25.8) $$\vec{F} = -\overrightarrow{grad} U = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right) \equiv -\left(\partial_x U, \partial_y U, \partial_z U \right) = -\vec{\nabla} U$$

25.1b Reminder of the potential energy function in gravity:

The potential energy U of x, y, and z for the gravitational force of a point-mass m inserted in the gravitational field created by M is:

(25.9) $$U(r) = \frac{-mMG}{r} = \frac{-mMG}{\sqrt{x^2 + y^2 + z^2}}; \text{ with the reference at } \infty; U(r \to \infty) = 0$$

One can check that according to (25.8) $F_x = -\partial_x U$

The x component of the force of gravity $\vec{F}$ is given by:

(25.10)
$$F_x = -\frac{\partial}{\partial x} U(r) = -\frac{\partial}{\partial x} \left(\frac{-mMG}{\sqrt{x^2 + y^2 + z^2}} \right) = \frac{mMG \left(-\frac{1}{2} \right) 2x}{\left(x^2 + y^2 + z^2 \right)^{\frac{3}{2}}} = \frac{-mMG}{\left(x^2 + y^2 + z^2 \right)^{\frac{3}{2}}} x$$

which is indeed the correct x component of F_g

Therefore,

(25.11) $$\vec{F}_g = \frac{-mMG}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\langle x,y,z\rangle = \frac{-mMG}{r^2}\vec{u}_r$$

Let us go back to electrostatics where we get the same relationship for charges in an electric field, with the exceptions that we have positive and negative charges.
The force on charge q_1 created by an electric field generated by charge q_2 is

(25.12) $$\vec{F} = kq_1q_2\frac{\vec{u}_r}{r^2} = \frac{kq_1q_2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\langle x,y,z\rangle ; \vec{F} = q_1\vec{E}$$

25.1c Potential energy and work:
The work done by a conservative force in moving a particle from point a to point b, is equal to the negative change in its potential energy. **(Remember that the work done by an outside agent on a particle without accelerating it is equal to +ΔU. Example: if you lift a rock of mass m through a vertical distance of h, the you do the positive work mgh, which is equal to the change of potential energy of the rock. The gravitational force and the gravitational field point from the higher potential energy location to the lower potential energy location.**)

(25.13) $$\boxed{\Delta U = -W = -\oint_{\substack{\text{along any path}\\ \text{from a to b}}} \vec{F}\bullet d\vec{s}}$$

This implies that the gradient operation on a conservative field is the inverse of the line integral operation. In the following notation of the gradient as the derivative of U with respect to the line element $d\vec{s}$ this is made more evident:
(25.14)

(25.15) $$dU = -\vec{F}\cdot d\vec{s} \Leftrightarrow \vec{F} = -\frac{dU}{d\vec{s}} \hat{=} -\overrightarrow{grad}\cdot U$$

(25.16) $$\boxed{\begin{array}{l}\vec{\nabla}\times\vec{F} = \vec{0} \Rightarrow \vec{F} = -\overrightarrow{grad}U \\ \vec{F}\ is\ \text{a conservative force, and U is its potential energy.}\end{array}}$$

It also means that the line integral of the force is 0 if the beginning and end points are the same, or, to say it with different words, if the line integral is taken over a (simple) closed loop.

25.1d Conservative Fields in Electrostatics:
In electrostatics, we ignore the electric effects of **moving charges**, and the force associated with a constant (in time) electric field is given by $\vec{F} = q_0\vec{E}$. As defined in the previous chapter the electrostatic field $\vec{E}(\vec{r})$ at a point in space is the resultant force of all electric charges at that

point acting on a positive test charge, divided by the test charge. **If follows that the electric field is also a conservative field, just like the associated force**:

(25.17)
$$\vec{F} = q_0\vec{E} \Rightarrow \vec{\nabla}\times\vec{F} = \vec{0} \Rightarrow \vec{\nabla}\times\vec{E} = \vec{0} \Rightarrow$$
$$\vec{E} = -\overrightarrow{grad}\frac{U}{q_0} \equiv -\overrightarrow{grad}V; E_x = -\frac{\partial V}{\partial x} \text{ and so on.}$$

The analogy holds of course also for the integral form of the relationship above:

(25.18)
$$\text{integral form: } \Delta V = -\int_a^b \vec{E}\cdot d\vec{s}$$
$$\text{differential form: } dV = -\vec{E}\cdot d\vec{s}$$

Electric field lines are directed from a higher potential V to a lower potential V, therefore always from V+ to V-. They are perpendicular to lines (or surfaces) of constant potential. More on this in a moment. (This is analogous to the direction of the gravitational field which points from a higher potential energy to a lower potential energy $\vec{\Gamma} = -\overrightarrow{grad}\cdot\Phi$. It also corresponds to the heat current density vector which points from a higher temperature to the lower temperature. $\vec{j}_h = -k\overrightarrow{grad}\cdot T$ and to the diffusion current $\vec{j}_n = -D\overrightarrow{grad}\cdot n$.)

We see that the **electric potential V** is equal to the **electric potential energy U** divided by the elementary charge q_0, which can be positive or negative. The electric potential V at any point in an electric field is given by

(25.19)
$$V = \frac{U}{q_0}, \text{ with reference to a point (often ground 0 of the earth)}$$
where the electric potential V is set equal to 0.

The change $\Delta V = V_2 - V_1$ of the electric potential between any two points in an electric field is equal to:

(25.20)
$$\Delta V \equiv \frac{\Delta U}{q_0} = V_b - V_a = -\int_a^b \underbrace{\vec{E}\cdot d\vec{s}}_{dV}$$
$$dV = -\vec{E}\cdot d\vec{s}$$

The dimension for the electric potential difference is equal to Joules/Coulomb which has the name volt. (in honor of the Italian physicist Volta). The electric potential difference is therefore also often called voltage difference or simply voltage.

(25.21)
$$1Volt = \frac{1Joule}{1Coulomb}$$

In the case of a **uniform electric field,** like between the plates of a parallel plate capacitor where the field is given by $\vec{E} = \frac{\sigma}{\varepsilon_0}\vec{u}$, the path independent line integral above becomes simply a scalar product between the uniform vector of the electric field and the distance d between a and b, which is the distance between the two plates of the capacitor.)

(25.22)
$$\Delta V = -\int_{\substack{path \text{ from one} \\ \text{plate to the other}}} \vec{E} \cdot d\vec{s} = Ed$$

Memorize:
Electric field lines always point in the direction of decreasing electric potential V.

(25.23)
$$\vec{E} = -\vec{\nabla} \cdot V \equiv -\overrightarrow{grad} \cdot V$$

From this we can also see that the dimension of the electric field can be conveniently expressed through Volt/meter:

(25.24)
$$[E] = \frac{[V]}{[d]} = \frac{Volt}{meter} = \frac{V}{m} = \frac{N}{C};$$
$$[\text{x}] \equiv \text{ dimension of x}$$

This is of course consistent with the fact that

(25.25)
$$\vec{E} = -\overrightarrow{grad}V \;\; E_x = -\frac{\partial V}{\partial x}$$ the value of the electric field is the rate of change of the voltage with respect to distance x.
$$E_x = -\underbrace{\lim}_{\Delta x \to 0} \frac{\Delta V}{\Delta x}$$

Now the unit of eV should become clearer also:
1electron Volt=1eV is the energy necessary to move an elementary charge e across a voltage difference of 1 Volt.

(25.26)
$$1eV = 1.6 \cdot 10^{-19} J$$
$$1MeV = 1\text{Mega-electronvolt}=10^6 eV = 1.6 \cdot 10^{-13} J$$

Let us also recall the **work energy theorem for conservative forces**:
$$\Delta U + \Delta K = 0 \Leftrightarrow U_1 + K_1 = U_2 + K_2$$

(25.27)

$$W = -\Delta U = -q_0 \Delta V = \int_a^b q_0 \vec{E} \cdot d\vec{s} = \Delta K = \frac{1}{2} m(\mathrm{v}_b^2 - \mathrm{v}_a^2)$$

For any conservative force an increase in potential energy means a decrease in kinetic energy and vice versa. $\Delta K + \underbrace{q_0 \Delta V}_{\Delta U} = 0$

Work done by the conservative force = $W = \Delta K = -\Delta U = -q_0 \Delta V$

Note here again, that if an **outside agent moves a positive charge without acceleration**, the work done by that agent is +ΔU.

(25.28) Work by outside agent (without acceleration): $\mathrm{W} = +\Delta U = +\mathrm{q}_0 \Delta V$

Recall the situation where a person lifts a rock of mass m by a distance y in the gravitational field on earth. The work done by gravity is $-mgy$, but the work done by the person is $+mgy$, which is equal to the potential energy of the rock at height y.
A positive charge placed into an electric field will be moved along the electric field lines in the direction from positive to negative. The charge will be accelerated by this force, according to

(25.29)
$$W = -q\Delta V = -\int_a^b q\vec{E} \cdot d\vec{s} = \Delta K$$

25.2 The Electrostatic Potential V.

If the force in question is an electrostatic force the sign of the potential energy also depends on the sign of the test charge q_0 which we put into the field. **If a positive charge, having a certain initial velocity, is moving against the electric field, the potential energy (voltage) will increase (the kinetic energy of the charge will decrease). If a positive charge is allowed to move with the electric field, its kinetic energy increases and therefore its potential energy decreases**. For a negative test charge the situation is reversed.

Example 1: Assume there are two equipotential field lines, one at 2 Volts, the other at 5 Volts. By definition, the potential does not change along these lines. There is an electric field at every point of these lines. **The direction of these fields is always perpendicular to the equipotential field lines**. Moving tangentially along an equipotential field line will yield a zero change in potential energy, dV=0. But $dV = -\vec{E} \cdot d\vec{s} = 0 \; if \; \vec{E} \perp d\vec{s}$

The scalar product between a non-zero electric field $\vec{E}$, and the tangential line element $d\vec{s}$ can only be 0 if the angle is 90°. Thus, the only possible direction for an electric field is perpendicular to both equipotential lines.
If an outside agent moves a positive charge of 1mC from V=2 Volts to V=5 Volts, the path against the direction of $\vec{E} = -\overrightarrow{grad} \cdot V$ requires the amount of work for the outside agent W=1mC·3Volts =3mJ. The outside agent does positive work and increases the potential energy of the system.

25.2a **Summary**:

- $dV = -\vec{E} \cdot d\vec{s} = 0$ when $\vec{E}$ is perpendicular to $d\vec{s}$, a line element along the eps.
- Lines for which dV=0 are called equipotential field-lines, or, in three dimensions, equipotential surfaces (eps).
- As $\vec{E} = -\overrightarrow{grad} \cdot V$ this means that $\vec{E} = -\overrightarrow{grad} \cdot V$ is also perpendicular to the equipotential surfaces. dV is a maximum if $d\vec{s}$ is parallel to $\vec{E}\ or - \overrightarrow{grad} \cdot V$ which is $\perp$ to the eps.
- Between any two points of a given eps with V_a and V_b we have $\Delta V = \text{constant} = -\int_a^b \vec{E} \bullet d\vec{s} = V_a - V_b$
- As there is only one electric field line possible connecting two points it must start at a direction perpendicular to the first equipotential surface in a given point and end at a perpendicular direction on the second eps.
- The magnitude of $-\overrightarrow{grad} \cdot V$ is the maximum change of the potential function and $\vec{E}$ points in the direction of decreasing values for V.
- We say that the electric field follows its gradient, which means that it follows the direction of maximum change of its potential field.
- We can more readily visualize this by considering energy conservation in the form $\Delta K + \Delta U = 0$. A charge does the minimum work when if follows its gradient. As a result its gain in kinetic energy is at a maximum.

Think of water running down a hill. It gains kinetic energy and loses potential energy.

$$\Delta K = -\Delta U \Rightarrow \frac{\Delta K}{q_0} = -\Delta V$$

Example 2: If the electric field vector $\vec{E}$ is parallel to the displacement $d\vec{s}$ we get the maximum change of the potential dV. If the perpendicular distance between two level curves of values 8 and 12 volts at a given point is 0.5 cm, we know that the maximum strength of the electric field is

(25.30)
$$\left|\vec{E}\right| = \frac{\Delta V}{\Delta s} = \frac{4Volts}{0.5cm} = 800\frac{V}{m}$$

The direction of the electric field is from the 12 volt level to the 8 volt level.

Example 3: A 12 V battery is connected to two parallel plates 0.30 cm apart. Find the potential difference and the electric field inside of this capacitor.

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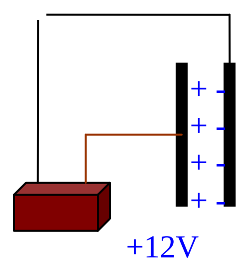
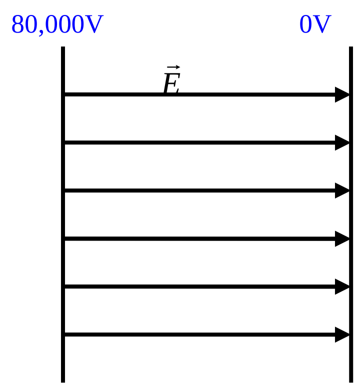
Figure 1



Well, ΔV=12 Volt. But ΔV=Ed, where d is the separation between the two plates. Therefore the electric field inside the capacitor is 4000 V/m. The field is directed from + to -. The negative plate is connected to the negative pole of the battery (cathode) and the positive plate is connected to the positive pole of the battery (anode). The battery pumps electrons onto the negative plate. They repel electrons on the opposite plate. These repelled electrons flow back into the battery, thus leaving a positively charged inside plate.

Example 4: A proton is released from rest in a uniform electric field with strength of 80,000 V/m. The proton moves by a distance of 0.50 m in the direction of $\vec{E}$.

Figure 2



a) Find the change in electric potential between the initial and final points. The proton will move in the direction of decreasing potential for a distance of 0.50m

$$\Delta V = -\int_i^f \vec{E} \bullet d\vec{s} = -Ed = -40,000V$$

The electric field exerts a positive force on the proton, which will accelerate and gain kinetic energy. As we have conservation of energy E=K+U the potential energy of the proton must decrease by the same amount as the kinetic energy increases.

b) Find the change in potential energy of the proton-field system.

(25.31) $$\Delta U = q_0 \Delta V = +1.6 \cdot 10^{-19} C \cdot (-40,000V) = -6.4 \cdot 10^{-15} J$$

c) Find the speed of the proton at the end of 0.50 m.

$$\Delta U + \Delta K = 0 \Rightarrow \Delta K = -\Delta U = 6.4 \cdot 10^{-15} J$$

(25.32) $$\text{v}=\sqrt{\frac{2}{\text{m}} 6.4 \cdot 10^{-15} J} = 2.8 \cdot 10^{6} \frac{m}{s}$$

(25.33)Remember:
$$Work = W = \int_{\textit{path} \text{ from a to}}^{\text{b}} \vec{F} \cdot d\vec{s} = -\Delta U = -q\Delta V \Rightarrow$$
$$\Delta V = -\int_{a\,\text{(any path)}}^{b} \vec{E} d\vec{s} = V_a - V_b$$

The electric field lines are perpendicular to the lines of constant electric potential. **The electric field points from a higher potential to a lower potential**.

(25.34) $$\vec{E} = -\overline{grad} \cdot V$$

25.3 Calculation of the Potential Energy U and the Electric Potential V, due to point-charges:

The electric field at location $\vec{r}$ of a point charge q located as (0,0,0) is given by:

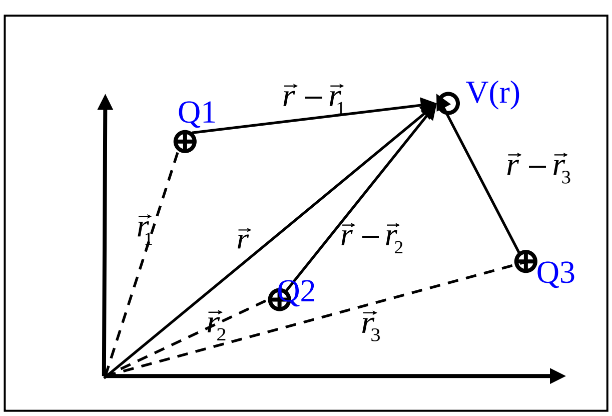
(25.35) $\vec{E} = \frac{k_e q}{r^2}\vec{u}_r$ compare to the gravitational field of mass M: $\vec{\Gamma} = -\frac{MG}{r^2}\vec{u}_r$

From our discussion about conservative fields and forces it follows that the electric potential difference can be calculated through the line integral between two points, given that the line integral depends only on the initial and final point. (This means that the line integral becomes a regular definite integral.) We then set the potential value at the initial or reference point equal to 0. As in the case of the gravitational potential energy, that point is taken at infinity.

(25.36) $$\Delta V = V_f - V_i = \Delta V = -\int_i^f \vec{E} \bullet d\vec{s} = -\int_i^f \frac{k_e q}{r^2}\vec{u}_r \bullet d\vec{s} = -\int_i^f \frac{k_e q}{r^2} dr = -(-)\left(\frac{k_e q}{r_f} - \frac{k_e q}{r_i}\right) = \frac{k_e q}{r_f} - \frac{k_e q}{r_i}$$

(25.37)
$$V_f \equiv V(r) = \frac{k_e q}{r} = \frac{k_e \text{q}}{\sqrt{x^2 + y^2 + z^2}} \text{ ; compare to } \Phi_g = -\frac{MG}{r}$$
with $V(r_i) = 0$ for $\text{r}_\text{i} \to \infty$

For a series of point charges q_i the potential function V(r) at a point r is the sum of all the potentials due to the individual charges q_i.



(25.38)
$$V(r)=k_e\sum_{i=1}^{n}\frac{q_i}{|\vec{r}-\vec{r}_i|}=k_e\sum_{i=1}^{n}\frac{q_i}{\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}};V(r)=0 \text{ for } r\to\infty$$

For charges located on the x-axis: $V(x)=k_e\sum_{i=1}^{n}\frac{q_i}{(x-x_i)};V(x)=0 \text{ for } x\to\infty$

$|\vec{r}-\vec{r}_i|=\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}$ is the distance from the point $\vec{r}$ where we calculate V(r) =V(x,y,z) and the point $\vec{r}_i$ which is the location of the individual charges q_i. **Note that the potential function is a scalar function, not a vector**.

We can always obtain the electric field by taking the derivative of the potential field:

(25.39) $$\vec{E}(x,y,z)=-\overrightarrow{grad}V(x,y,z)$$

(25.40)

$$\vec{E}(x,y,z)=-\overrightarrow{grad}\left(k_e\sum_{i=1}^{n}\frac{q_i}{\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}}\right)$$

$$=-\frac{\partial}{\partial x}\left(k_e\sum_{i=1}^{n}\frac{q_i}{\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}}\right)=-k_e\sum_{i=1}^{n}q_i\frac{\left(-\frac{1}{2}\right)2(x-x_i)}{\left((x-x_i)^2+(y-y_i)^2+(z-z_i)^2\right)^{\frac{3}{2}}}=$$

$$E_x(x,y,z)=k_e\sum_{i=1}^{n}q_i\frac{(x-x_i)}{\left((x-x_i)^2+(y-y_i)^2+(z-z_i)^2\right)^{\frac{3}{2}}}$$

The results for the y and z components of the electric field vector are obtained in the same way.

Potential Energy of a System of Charges:

In order to calculate the potential energy of a system of charges one has to consider how this system is brought together. For example: If three charges form an original system, they have an electric potential V(r) at the point $\vec{r}$. If a fourth charge q_4 is brought into this system, we get a different potential energy $U=q_4V(r)$ from when all four charges are assembled from infinity

$U=k_e\sum_{i,j=1}^{4}\frac{q_iq_j}{|\vec{r}_i-\vec{r}_j|};i<j$ (See (25.43)). Let us start with two charges:

25.3a Electrostatic Potential Energy of point charges.

The **potential energy** of a system of 2 point charges at a distance r_{12} $=|\vec{r}_1-\vec{r}_2|$ from each other is the energy needed **for an outside agent** to bring charge q_2 from infinity to a distance r_{12} from the potential of charge q_1, without acceleration (Compare to bringing a mass m into the gravitational field of the sun with mass M, or lifting a mass m by a distance h, gives the mass a potential energy of mgh.). Note that this potential energy is not a function of a location. It is a potential difference with the 0 reference point at infinity.

(25.41)
$$U = q_2 V_1 = q_2 \frac{kq_1}{r_{12}} = q_2 \frac{kq_1}{|\vec{r}_1 - \vec{r}_2|}$$

Note that if the charges are of the same sign, U is positive. If the charges are of opposite signs U is negative (attraction). If they are of the same sign (repulsion), an outside agent must do positive work to overcome the repelling force between the charges.

The potential energy of the system of three charges is:

(25.42)

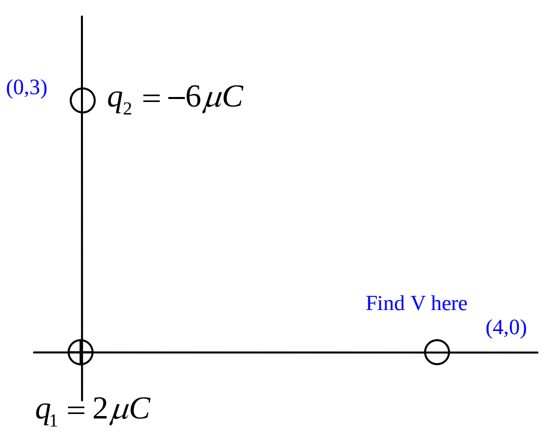
$$U = k_e \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right) = k_e \left(\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|} + \frac{q_1 q_3}{|\vec{r}_1 - \vec{r}_3|} + \frac{q_2 q_3}{|\vec{r}_2 - \vec{r}_3|} \right);$$ the charges can be positive or negative.

For a system of more than three charges we must add the contributions of all the pairs, **without duplication.** This means that we use only one of any two possible combinations because $r_{ij}=r_{ji}$:

(25.43)
$$\Delta U = U = k_e \sum_{i,j} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}; i < j$$

For 5 different charges we get the following valid combinations: 12,13,14,15,23,24,25,34,35,45.

Example 5: A charge q_1=2μC is located at the origin, and a charge q_2=-6μC is located at (0,3)m. Find the total electric potential due to these charges at the point (4,0)m.



From (25.38) we get:

$$V(r)=k_e\sum_{i=1}^{n}\frac{q_i}{|\vec r-\vec r_i|}=V(4,0)=k_e\sum_{i=1}^{2}\frac{q_i}{|\vec r-\vec r_i|}=k_e\left(\frac{2\mu C}{4}+\frac{-6\mu C}{5}\right)=-6.29kV$$

$$|\vec r-\vec r_i|=\sqrt{(x-x_i)^2+(y-y_i)^2}$$

Now, find the change in potential energy of this system of two charges plus a charge q_3=3μC as the latter charge moves from infinity to the point r.

(25.44) $$\Delta U=q_3V_{12}=3\mu C\cdot(-6.29kV)=-1.89\cdot10^{-2}J$$

Now, what would be the change in potential energy (what would be the energy of the system of the three charges) when all three charges start out infinitely far apart?
In this case we have to calculate all contributions of all the charges according to (25.43)

(25.45) $$\Delta U=U=k_e\left(\frac{q_1q_2}{|\vec r_1-\vec r_2|}+\frac{q_1q_3}{|\vec r_1-\vec r_3|}+\frac{q_2q_3}{|\vec r_2-\vec r_3|}\right)=k_e\left(\frac{-12}{3}+\frac{6}{4}+\frac{-18}{5}\right)\frac{(\mu C)^2}{m}=-5.48\bullet10^{-2}J$$

25.4 Obtaining the value of the electric field from the electric potential V.

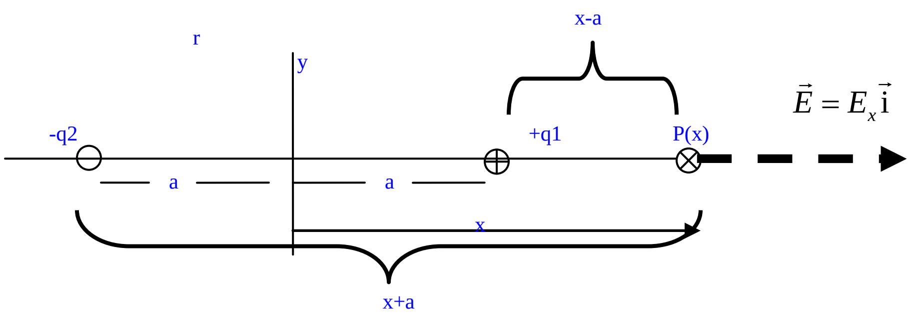
It is usually easier to calculate the electric potential (a scalar) than the electric field (a vector). We know the general relationship is:

$$\vec E=-\overrightarrow{grad}\cdot V$$

(25.46) $$\vec\nabla=\left\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\rangle \text{ in cartesian coordinates}$$

$$\vec\nabla=\frac{\partial}{\partial r}\vec u_r \text{ for a spherical symmetric situation}$$

(no angular dependency)



As an example, **find the electric field due to a dipole**, by calculating V(r) first:
Let us first calculate the electric potential of the dipole at the point x on the positive x axis to the right of x=a. We position the positive charge q_1 at x_1=a and the negative charge at x_1=-a. We notice that at point x the electric field E_1 due to the positive charge q_1 is larger than the electric field due to the charge q_2. Therefore the resultant electric field will point to the right. It also means that the potential V(x) is positive.

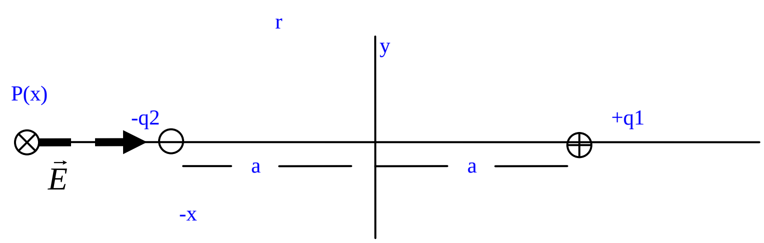
From: (25.38) $V(r)=k_e\sum_{i=1}^{n}\frac{q_i}{|\vec{r}-\vec{r}_i|}; V(r=\infty)=0$ reference point we get: $V(x)=k_e\sum_{i=1}^{n}\frac{q_i}{|x-x_i|}; V(y)=0$

(25.47)
$$V(x)=k_e\left(\frac{q_1}{|x-x_1|}+\frac{q_2}{|x-x_2|}\right)=k_e\left(\frac{q}{x-a}+\frac{-q}{x+a}\right)=k_eq\left(\frac{1}{x-a}-\frac{1}{x+a}\right)=$$
$$V(x)=k_eq(\frac{x+a-(x-a)}{x^2-a^2})=k_eq\frac{2a}{x^2-a^2}$$

We get the electric field by taking the negative gradient, which in this one-dimensional case reduces itself to the x-derivative:

(25.48)
$$E_x(x)=-\frac{dV}{dx}=2k_eqa\left(\frac{2x}{\left(x^2-a^2\right)^2}\right)=\frac{4k_eqax}{\left(x^2-a^2\right)^2}$$

If the point x is located to the left of charge q_2 the electric field E_2 due to the negative charge is



larger than the electric field due to the positive charge is farther away from P. This means that the resultant field will again point to the right.

The differences in the denominators must be the positive distances between the charge and the location where we calculate the potential function V: In the above example, the negative charge is closer to the point where we calculate the potential function than the positive charge. Therefore the sum must turn out negative.

$$V(x)=k_e\left(\frac{q_1}{|x-x_1|}+\frac{q_2}{|x-x_2|}\right)=k_e\left(\frac{q}{x+a}+\frac{-q}{x-a}\right)=k_eq\left(\frac{1}{x+a}-\frac{1}{x-a}\right)=$$
$$k_eq\left(\frac{x-a-(x+a)}{x^2-a^2}\right)=\frac{-2k_eqa}{x^2-a^2}$$
(25.49)

For the electric field we expect again a positive x component:

(25.50) $$E_x(x) = -\frac{\partial V}{\partial x} = \frac{2k_e q a 2x}{\left(x^2 - a^2\right)^2} = \frac{2k_e p x}{\left(x^2 - a^2\right)^2}; p \equiv 2aq$$

For x=0 we get a 0-potential field, but a negative electric field (by inspection):

$$E_x\left(x = 0\right) = \frac{-2k_e q}{a^2}$$

For distances on the x-axis much larger than the distance between the two charges of the dipole, we can neglect the distance "a" in the denominator. We also use the conventional notation p=2aq for the magnitude of the dipole moment:

(25.51) Dipole moment p directed from the negative to the positive charge: $\vec{p} = 2\vec{a}q$

We get for the electric field of the dipole, with the negative charge to the left of the positive charge, at long distances from the charges, along the direction of the dipole moment:

$$\vec{E}\left(x\right) = \frac{2k_e p}{\left|x\right|^3}\vec{i}$$

<u>25.4a Calculation of the potential function of a dipole in polar coordinates</u>:

For an arbitrary point in the x-y plane, we get the electrostatic potential V in polar coordinates as an approximation:

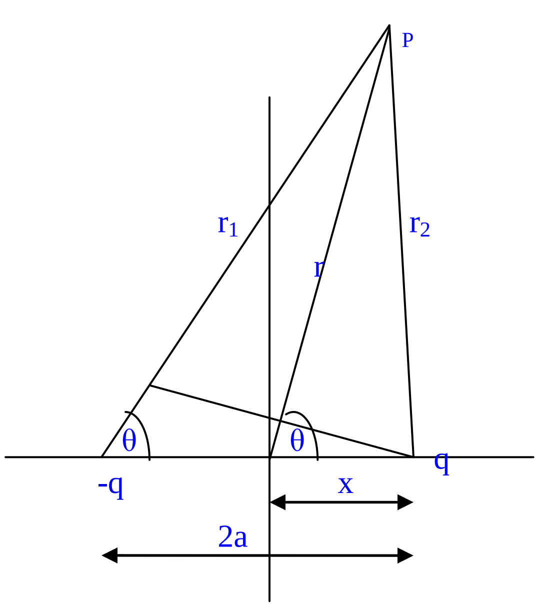
$$V(x,y) = k_e\left(\frac{q_1}{r_1} + \frac{q_2}{r_2}\right) = k_e q\left(\frac{-1}{r_1} + \frac{1}{r_2}\right) =$$

(25.52) $$= k_e q\left(\frac{-r_2 + r_1}{r_1 r_2}\right) \approx k_e q\left(\frac{2a\cos\theta}{r^2}\right)$$

$$V(r,\theta) = 2aq\frac{k_e\cos\theta}{r^2}$$

with $r_1 \cong r_2 \cong r$ and $r_1 - r_2 \cong 2a\cos\theta$

For <u>very long distances r the three radii are parallel</u>, and the triangle with angle θ is a right triangle.



The product of the distance between the two charges and the charge is often called the dipole moment: p=2aq. $\vec{p} = \vec{d}\cdot q$ The vector $\vec{d}$ points from the negative to the positive charge and has as its length the distance between them.

(25.53) $$V = k_e p\frac{\cos\theta}{r^2}$$

To calculate the electric field by taking the gradient of V, we express our variables in terms of x, y, and z . For the plane, we use x and y variables only. $\cos\theta \approx \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$

(25.54)
$$r = \sqrt{x^2+y^2};\ \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$
$$V(x,y) = k_e p \frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}}$$

We can easily obtain the components of the electric field from (25.54) by using $\vec{E} = -\overrightarrow{grad}\cdot V$. The value of the electric field on the x-axis is consistent with the result (25.48). We can obtain the electric field anywhere in the x-y plane, as long as the location is far away from the dipole.

(25.55)
$$E_x = -\frac{\partial V}{\partial x} = -\frac{\partial}{\partial x}V(x,y) = -k_e p \frac{\partial}{\partial x}\left[\frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}}\right] = -k_e p\left[\frac{1}{\left(x^2+y^2\right)^{1.5}} - 1.5\frac{2x^2}{\left(x^2+y^2\right)^{2.5}}\right] =$$
$$E_x = -k_e p \frac{x^2+y^2-3x^2}{\left(x^2+y^2\right)^{2.5}} = \frac{k_e p\left(2x^2-y^2\right)}{\left(x^2+y^2\right)^{2.5}}$$

(25.56)
$$E_y = -\frac{\partial V}{\partial y} = -\frac{\partial}{\partial y}V(x,y) = -k_e p \frac{\partial}{\partial y}\left[\frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}}\right] = \frac{3k_e p}{2}\left(\frac{2xy}{\left(x^2+y^2\right)^{2.5}}\right)$$

These approximations are very good as long the dipole length is very small in comparison with r. If the dipole is created by a molecule the length is around 10 nanometers. Any macroscopic distance would justify these approximations.

By the way, our calculations can also be performed using **polar coordinates for the gradient operator**

(25.57)
$$\boxed{\overrightarrow{grad}\cdot V\left(r,\theta\right) = \frac{\partial V}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{u}_\theta}$$

(25.58)
$$V\left(r,\theta\right) = k_e p \frac{\cos\theta}{r^2}$$

This yields the electric field in polar coordinates:

(25.59)
$$\vec{E}(r,\theta) = -\overrightarrow{grad}\cdot V$$

(25.60)
$$E_r = -\frac{\partial V}{\partial r} = -\frac{\partial}{\partial r}\left(k_e p \frac{\cos\theta}{r^2}\right) = +k_e p \frac{\cos\theta}{r^3}$$
$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -\frac{1}{r}\frac{\partial}{\partial \theta}\left(k_e p \frac{\cos\theta}{r^2}\right) = \frac{k_e p \sin\theta}{r^3}$$

25.5 Electric Potential Due to Continuous Charge Distributions.

For a continuous charge distribution the sum in equation (25.38) becomes an integral:

(25.61)

$$dV = \frac{k_e dq}{r} \Rightarrow V(P) = \int \frac{k_e dq}{r} \Rightarrow V(r) = \int_{\substack{\text{charged}\\ \text{body (1)}}} k_e \frac{dq(1)}{|\vec{r} - \vec{r}_1|};$$

$$V(x,y,z) = \int_{\substack{\text{charged body}\\ \text{with variables}\\ x_1, y_1, z_1}} \frac{k_e \rho_1 dV_1}{\sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}}$$

Note: V(r)=V(P)=V(x,y,z)

Note that the distances do not involve directions: V is a scalar function. To avoid confusion, distinguish between the coordinates at which we calculate the potential function (x,y,z) and the domain over which we integrate to obtain the charge (x_1,y_1,z_1) or some other variables over which we integrate. These variables must disappear in the final result.
If the electric field is already known we can use the equation (25.62)

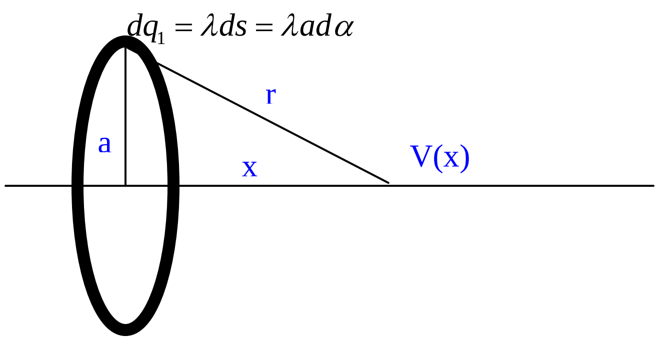
(25.62)

$$\text{from } \vec{E} = -\overrightarrow{grad} \cdot V \Rightarrow \Delta V = -\int_A^B \vec{E} \bullet d\vec{s}$$

to calculate the potential difference, and then choose a convenient 0 reference point.

25.5a Potential of a charged ring. Example 6:

Find the electric potential at a point P located on the perpendicular axis of a uniformly charged ring of radius a and total charge Q.



We see that the distances are the same for every point on the ring, therefore the integration does not involve the distance r but only the angle α.

$$V(x) = \int \frac{k_e dq_1}{r} = k_e \frac{\lambda a}{r} \int_{\alpha=0}^{\alpha=2\pi} d\alpha = \frac{k_e 2\pi a \lambda}{\sqrt{x^2 + a^2}} = V(x)$$

$$= \frac{k_e Q}{\sqrt{x^2 + a^2}}$$

We get the electric field by taking the derivative with respect to x:

$$E_x = -\frac{\partial V}{\partial x} = \frac{1}{2} k_e Q \frac{2x}{(x^2 + a^2)^{\frac{3}{2}}}$$

$$= k_e Q \frac{x}{(x^2 + a^2)^{\frac{3}{2}}}$$

We can also get the electric field with respect to r by using the polar form of the gradient. Conveniently, the "r" component of the gradient is simply: $-\frac{\partial}{\partial r}$ V(r).

Therefore we get: $E(r) = -\frac{\partial}{\partial r}\frac{k_e Q}{r} = \frac{k_e Q}{r^2}$ As we can see from the picture, this is the magnitude of the electric field at the point x. The x-component can be obtained by multiplying with the cosine of the angle between r and x, namely: $\frac{x}{r}$

25.5b Potential of a charged disk; Example 7:
For a uniformly charged disk with radius R we have to use the previous expression of V as our infinitesimal potential function dV and integrate over the variable radius "a" from 0 to R.

$$dq = 2\pi a\sigma da \text{ and } V=\sigma\pi\int_{a=0}^{a=R}\frac{k_e 2ada}{\sqrt{x^2+a^2}}$$

We make the substitution for the radicand

$$V=\sigma k_e\pi\int_{a=0}^{a=R}\frac{2ada}{\sqrt{x^2+a^2}}; x^2+a^2=u \Rightarrow 2ada=du \Rightarrow \sigma k_e\pi\int_{a=0}^{a=R}\frac{du}{u^{\frac{1}{2}}} = \sigma k_e\pi 2u^{\frac{1}{2}}\Big| =$$

(25.63)

$$V(x) = 2\sigma k_e\pi\left(\sqrt{x^2+R^2}-x\right)$$

For the electric field component E_x we get:

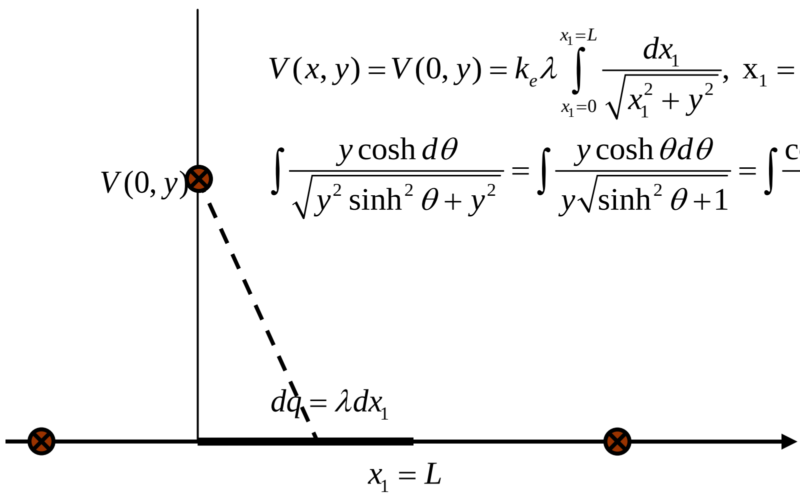
(25.64)
$$E_x = -\frac{\partial}{\partial x}V(x) = 2\sigma k_e\pi\frac{\partial}{\partial x}\left(x-\sqrt{x^2+a^2}\right) = 2\sigma k_e\pi\left(1-\frac{x}{\sqrt{x^2+a^2}}\right)$$

25.5c Potential of a charged line.

Example 8: Find the potential field V(0,y) of a line of charge. The line has length L and starts at x=0.

$$V(x,y) = V(0,y) = k_e\lambda\int_{x_1=0}^{x_1=L}\frac{dx_1}{\sqrt{x_1^2+y^2}},\ x_1 = y\sinh\theta, dx_1 = y\cosh\theta d\theta$$

$$\int\frac{y\cosh d\theta}{\sqrt{y^2\sinh^2\theta+y^2}} = \int\frac{y\cosh\theta d\theta}{y\sqrt{\sinh^2\theta+1}} = \int\frac{\cosh\theta d\theta}{\cosh\theta} = \theta$$



$$\text{Useful: } \cosh^2 x - \sinh^2 x = 1 \Leftrightarrow \sinh^2 x + 1 = \cosh^2 x$$

(25.65) $$\cosh x = \frac{e^x + e^{-x}}{2}; \sinh x = \frac{e^x - e^{-x}}{2}$$
$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}); \cosh^{-1} x = \ln(x + \sqrt{x^2-1})$$

(25.66) $$dV(y) = \frac{k_e \lambda dx_1}{\sqrt{x_1^2 + y^2}}; \text{we integrate } x_1 \text{ from 0 to L}$$

$$V(y) = k_e \lambda \int_0^L \frac{dx_1}{\sqrt{x_1^2 + y^2}} \text{ the integral is given by:}$$

(25.67) $$\frac{x_1}{y} = \sinh\theta; \theta = Ar\sinh\frac{x_1}{y} = \ln\left(\frac{x_1}{y} + \sqrt{\frac{x_1^2}{y^2} + 1}\right) = \ln\left(\frac{x_1}{y} + \sqrt{\frac{x_1^2 + y^2}{y^2}}\right) = \ln\left(\frac{x_1}{y} + \frac{1}{y}\sqrt{x_1^2 + y^2}\right) =$$
$$V(y) = k_e \lambda \ln\left(\frac{x_1 + \sqrt{x_1^2 + y^2}}{y}\right)\Bigg|_0^L = k_e \lambda \ln\left(\frac{L + \sqrt{L^2 + y^2}}{y}\right) = \frac{k_e Q}{L} \ln\left(\frac{L + \sqrt{L^2 + y^2}}{y}\right)$$

If we want to calculate V(x) with x>L we get:

(25.68) $$dV(x) = \frac{k_e \lambda dx_1}{\sqrt{(x - x_1)^2 + 0}} = \frac{k_e \lambda dx_1}{x - x_1}; \text{we integrate } x_1 \text{ from 0 to L}$$
$$V(x) = -k_e \lambda \left[\ln \frac{x - L}{x}\right]$$

We get the electric field $E_x(x)$ by using the gradient:

$$E_x(x) = -\frac{\partial}{\partial x} V(x) = +k_e \lambda \frac{\partial}{\partial x}\left[\ln\frac{x-L}{x}\right] = k_e \lambda \left[\frac{1}{x-L} - \frac{1}{x}\right] =$$
$$k_e \lambda \left[\frac{x - (x-L)}{x(x-L)}\right] = k_e \lambda \left(\frac{L}{x(x-L)}\right)$$

This field is positive and the vector points to the right as expected.
If the point is located at a distance d from the charged line, x=d+L and x-L=d. We get

(25.69) $$E_x = k_e \lambda \left(\frac{L}{(d+L)d}\right)$$

If we want to calculate the potential field to a point to the left of the origin on the x-axis (x=-d), we know that we must get the same potential field at a point d from the end of the charged line. We must also get the same electric field magnitude, but E_x points to the left. The point x=L+d to the right, corresponds to a point x=-d to the left.

(25.70) $$V(-x) = -k_e \lambda \left[\ln\frac{-x - L}{-x}\right] = -k_e \lambda \left[\ln \frac{x + L}{x}\right]$$
$$E(-x) = -\frac{\partial V(-x)}{\partial x} = +k_e \lambda \left[\frac{1}{x+L} - \frac{1}{x}\right] = k_e \lambda \frac{x - (x+L)}{x(x+L)} = k_e \lambda \frac{-L}{x(x+L)} = -k_e \lambda \frac{L}{x(x+L)}$$

If we put our point at x=-d, i.e. at the distance d to the left of the origin, we use x=d in our formula for E(-x) we get:

(25.71) $$E_x = -k_e\lambda\left(\frac{L}{(d+L)d}\right)$$

25.5d Potential of a uniformly charge sphere: Example 9:

Electric potential due to a uniformly charged sphere. The charges are uniformly distributed throughout this non-conducting sphere:

The potential of a uniformly charged sphere is $\frac{k_e Q}{r}$ on the outside of the sphere. We calculate the inside potential by first calculating the electric field:

The electric field inside of the uniformly charged sphere was obtained by using Gauss law.

(25.72) $$E4\pi r^2 = \frac{\rho}{\varepsilon_0}\frac{4\pi}{3}r^3 \Rightarrow E(r) = \frac{\rho}{3\varepsilon_0}r \text{ for r} \le \text{R and } \rho = \frac{3Q}{4\pi R^3}$$

As we know the value of V on the sphere to be kQ/R, we can use a point on the surface as our reference point and calculate V inside of the sphere at a point D by using:

(25.73) $$\Delta V = -\int_i^f \vec{E}\bullet d\vec{s} = -\frac{\rho}{3\varepsilon_0}\int_R^D r dr = -\frac{\rho}{3\varepsilon_0}\frac{1}{2}\left(r_D^2 - R^2\right) = V(r_D) - V(R) = \frac{\rho}{\varepsilon_0}\frac{1}{6}\left(R^2 - r_D^2\right)$$

(25.74) $$V(r_D) = \frac{\rho}{\varepsilon_0}\frac{1}{6}\left(R^2 - r_D^2\right) + V(R)$$

(25.75) $$V(R) = \frac{k_e Q}{R} = k_e\rho\frac{4\pi}{3R}R^3 = \frac{4\pi}{3\cdot 4\pi\varepsilon_0}\rho R^2 = \frac{\rho R^2}{3\varepsilon_0}$$

(25.76) $$V(r_D) = \frac{\rho}{\varepsilon_0}\frac{1}{6}\left(R^2 - r_D^2\right) + \frac{\rho R^2}{3\varepsilon_0} = \frac{\rho}{6\varepsilon_0}\left(3R^2 - r_D^2\right)$$

25.5e Electric Potential Due To a Charged Conductor.

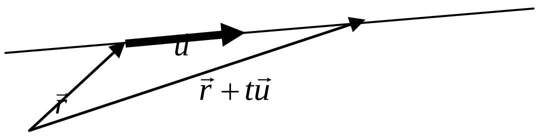
The surface of any charged conductor in electrostatic equilibrium is an equipotential surface. Because the **electric field** is 0 inside the conductor, the **electric potential** is constant everywhere inside the conductor and equal to its potential on the surface. There cannot be a different potential inside, because any difference to a potential would require the existence of an electric field. As there is no electric field inside the conductor, the potential field must be the same as the field on the surface.

For example, the potential of a charged conducting sphere is equal to $\frac{k_e Q}{R}$ on its surface and has therefore the same value on the inside.

25.6 Optional: Directional derivative and gradient:

Review of Vector Operators and Related Topics from a Physics Perspective:

Let $\vec{u}$ be a unit vector and $T(\vec{r})$ a scalar field like the temperature field, i.e. each point in space has a temperature assigned to it. We can define a function which tells us how the temperature field is changing in any direction defined by the unit vector:



(1.1) $\frac{\partial T}{\partial \vec{u}} = \lim_{\Delta t \to 0} \frac{T(\vec{r} + \Delta t \cdot \vec{u}) - T(\vec{r})}{\Delta t}$

The change of a scalar function is given by the gradient of that function. For a temperature field, we have:

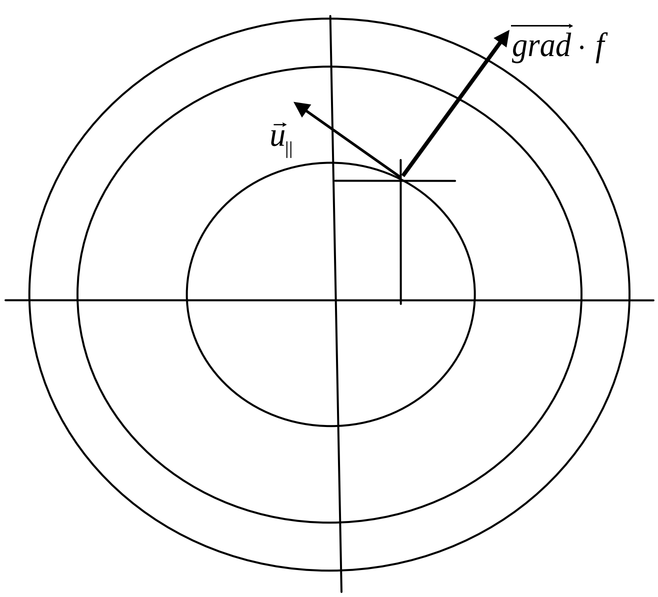
(1.2) $\vec{j} = -k\,\overrightarrow{grad}T$ or $\vec{E} = -\overrightarrow{grad} \cdot V$

where k is simply the conductivity constant. Multiplying this function by the unit vector gives us the scalar quantity of change of temperature or the potential in a particular direction.

(1.3) $\vec{j} \cdot \vec{u} = -k\,\overrightarrow{grad}T \cdot \vec{u}$ or $\vec{E} \cdot \vec{u} = -\overrightarrow{grad} \cdot V \cdot \vec{u}$

This is the current density multiplied by the direction. We can easily see now that **this flow is a maximum when the unit vector is parallel to the direction of the current density**. The set of points for which T is constant defines the **level curves (equipotential curves) or level surfaces (equipotential surfaces)**.

Below, we have three level curves in the xy plane. They are constant circles, for example $x^2 + y^2 = a^2, b^2, c^2$ with three different constant values. The corresponding function would be $f(x, y) = x^2 + y^2$ These are concentric circles with radii a, b and c. The gradient of this function in the point x=2 and y=3 is given by:



(1.4) $\overrightarrow{grad} \cdot f(x, y) = 2x\vec{i} + 2y\vec{j} \Rightarrow 4\vec{i} + 6\vec{j}$

The maximum change of the function f in any point is in the direction of the gradient.

The scalar product between a unit vector parallel to the level curve $\vec{u}_{\parallel}$ and the gradient is 0. $\vec{\nabla} f \cdot \vec{u}_{\parallel} = 0$ which means that the gradient of the scalar field is perpendicular to the level surface. If the unit vector is perpendicular to such a level curve the quantity above is the **normal** derivative

$\frac{\partial f}{\partial \vec{n}} \equiv (\overrightarrow{grad} \cdot f)\vec{n}$

This gives us the natural flow of a heat current which follows the **negative gradient** of T. The same is true for the flow of water down a hill, it follows the negative gradient.

This is also true, as we have seen for the electric field $\vec{E}$ which follows the negative gradient of the potential V. $\vec{E} = -\overrightarrow{grad} \cdot V$

Another example is the series of level curves x^2-y^2=A, which is a series of hyperbolas, representing the level curves of four equal charges.

Example: Suppose that the temperature at the point (x,y,z) is given by (1.5)

$T(x, y, z) = x^2 - y^2 + xyz + 273$

In which direction is the temperature increasing most rapidly? That would be in the direction of the gradient.

(1.6) $$\vec{\nabla} T = (2x + yz)\vec{i} + (-2y + xz)\vec{j} + xy\vec{k}$$

At the point (-1,2,3) this is equal to $\vec{\nabla} T(-1,2,3) = 4\vec{i} - 7\vec{j} - 2\vec{k}$

The rate of increase is $\left|\overrightarrow{grad}T\right| = \sqrt{16+49+4} = \sqrt{69}$ We know of course that the temperature is decreasing most rapidly in the direction of $-\overrightarrow{grad}T$ according to the formula for heat flow (See chapter 20) $\vec{j} = -k\,\overrightarrow{grad}T = -k\vec{\nabla} \cdot T$

Optional 25.7 Laplace and Poisson equation for the gravitational and electric fields.
(optional)

For both the gravitational field and for the electrostatic field we have the local expression of the divergence:

(1.7) $$div\vec{\Gamma} = -4\pi G\rho_g \text{ and } div\vec{E} = \frac{\rho_{el}}{\varepsilon_0}$$

We also know that both the gravitational field and the electrostatic field are conservative, thus both derive from a potential scalar field function. In order to distinguish them here I use the letter Φ for the gravitational potential and V for the electrostatic potential:

(1.8) $$\begin{aligned} \vec{\Gamma} &= -\vec{\nabla}\Phi \text{ or } \vec{\Gamma} = -\overrightarrow{grad}\ \Phi \\ \vec{E} &= -\vec{\nabla}V \text{ or } \vec{E} = -\overrightarrow{grad} \cdot V \end{aligned}$$

We see therefore that in both cases the vector fields can be expressed by the scalar fields:

(1.9) $$\begin{aligned} div\left(-\overrightarrow{grad}\ \Phi\right) &= -4\pi G\rho_g \ \Rightarrow \vec{\nabla}\cdot\vec{\nabla}\Phi = 4\pi G\rho_g \\ div\left(-\overrightarrow{grad}\cdot V\right) &= \frac{\rho_{el}}{\varepsilon_0} \Rightarrow \vec{\nabla}\cdot\vec{\nabla}V = -\frac{\rho_{el}}{\varepsilon_0} \end{aligned}$$

We have encountered such expressions before, This is the so-called Laplace operator, which occurs in many

(1.10) partial differential equations:

$$div \cdot grad = \vec{\nabla} \cdot \vec{\nabla} = \Delta = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The electrostatic potential obeys the typical Laplace equation:

(1.11)
$$\Delta V = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V = -\frac{\rho_{el}}{\varepsilon_0}$$

If the right side of this differential equation is 0, it is known as the Laplace equation which, with proper boundary conditions leads to description of the electrostatic potential in a particular domain which does not contain any charges.
If the charge density is not 0, the equation is known as the **Poisson equation**.
We see that similar differential equations appear in completely different physical contexts. The mathematical problems remain the same, and the solutions which may be easier to come by and interpret in one area can then be readily transferred into another area.
From our previous considerations we know that the electric potential at point (1) created by a charge at point (2) is given by:

(1.12)
$$V(r_1) = \frac{k_e q(2)}{|\vec{r}_1 - \vec{r}_2|} = \frac{k_e \mathrm{q}(2)}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}}$$

We can readily translate all this information from heat flow and temperature fields into electrostatics where we are dealing with electric fields (vector fields) and potential fields (scalar fields) in the exact same way.

(1.13)
$$\boxed{\vec{E} = -\overrightarrow{grad} \cdot V}$$