Homework: See webpage.

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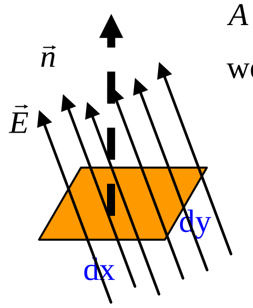
24.1 Flux of a vector field through a surface:

We can visualize the electric field as continuous lines, starting in a positive charge and ending in a negative one, or at infinity.The electric field vector-function is tangent to the electric field line at each point.The direction (arrow) of the line indicates the direction of the electric force on a positive test charge in the field.
The number of lines per unit area through a perpendicular surface is proportional to the electric field strength (its magnitude). Therefore the field lines are closer together in a stronger field.
Electric field lines can never cross, but can end in infinity, which is another way of saying that the negative charges are too far away to be considered.

A visual representation of a vector field $\vec{E} = \overrightarrow{E(x,y,z)}$ by field lines is very much similar to the stream lines of a liquid density, in which case we have encountered the liquid density vector $\vec{j} = \rho\vec{v}$ (ρ is the liquid density, and $\vec{v}(x,y,z)$ is the velocity vector field, which by definition can have a different value from location to location. See **ch20 First Law of Thermodynamics.doc**)
In the case of the electric field there is no flow of matter involved. It is a mathematical model. In the case of the intensity of a wave we encountered the concept of an **energy flow** through a cross-section. This concept is generalized with the concept of a flux Φ.
We define an **oriented surface** $\vec{A}$ *or* $\vec{S}$ as a **vector whose direction is perpendicular (normal) to a surface element and with the magnitude which is equal to the area**:

Figure 1



(24.1)

$\vec{A} = A \cdot \vec{n}$, $\vec{n}$ is a unit vector normal to the surface. For a small area we have $d\vec{A} = \vec{n}dA = \vec{n}dxdy$

(Recall that you can mathematically define a vector perpendicular to two other vectors in a plane by forming the cross-product of the two vectors in the plane.)

For a closed surface we define the normal direction as pointing to the outside of the enclosed volume. (A simply connected closed surface has an **inside** and an **outside** domain.)

The infinitesimal flux through a surface is defined for the electric field vector as

(24.2) $$d\Phi = \vec{E} \bullet d\vec{A} = E \cdot dA \cdot \cos\theta$$

this is the scalar product between the vectors $\vec{E}$ and $d\vec{A}$

For a surface we must integrate over the whole finite surface, thus defining the flux of the vector field $\vec{E}$ as the surface integral:

(24.3) $$\Phi_E = \iint_{surfaceA} \vec{E} \cdot d\vec{A}$$

Note that the above integral describes the flux through a single surface which is not closed. The cases we are most interested involve a closed surface, containing a volume inside of the surface.

For the flux through a closed surface we write:

(24.4) $$\Phi_{total} = \oiint_{closed\ surface} \vec{E} \cdot d\vec{A}$$

The actual integrals can be quite cumbersome to calculate. We use them only in cases with a nice symmetry, where the integration can be performed easily. So, don't be overwhelmed by the look of the formulas. For example, the flux of the **constant** vector $\vec{E} = E \cdot \vec{i}$ through the surface of a cube parallel to the y-z plane with area $\Delta\vec{A} = \Delta y \cdot \Delta z \cdot \vec{i}$ would be equal to $E\Delta x \cdot \Delta y$. The total flux through the whole cube of side-length s would be $-s^2E + s^2E = 0$.

We use such a simple situation to illustrate the basic principle underlying Gauss's law which is very useful in determining the electric field.

Example: Assume that we have an electric field in the x direction $\vec{E} = E_x\vec{i}$. We put a surface in the form of a cube with sidelength d in its path so that a side is in line with the direction of the electric field. We put the corner of the cube into the origin of the coordinate system (0,0,0). Both surfaces have the same area magnitude $A=d^2$. The area to the left has its normal unit vector pointing in the direction $-\vec{i}$. We call the two surfaces A_{in} on the left and A_{out} on the right. If $\vec{E}$ is uniform (constant with respect ot all variables x, y, z) the total flux is 0

(24.5) $$\Phi_{total} = \vec{A} \cdot \vec{E} = \left(\vec{A}_{in} + \vec{A}_{out}\right) \cdot \vec{E} = \left(-A\vec{i} + A\vec{i}\right) E_x\vec{i} = -AE + AE = 0$$

$-Ed^2+Ed^2=0$. This is simply the product between the magnitude of the field and the cross-sectional area. The normal vector to the surface is opposite to the entering E-field on the left which gives –, it is parallel to the leaving E-field on the right, $+Ed^2$. We can see that as long as the electric field does not change between the in-surface and the out-surface the total flux is 0.

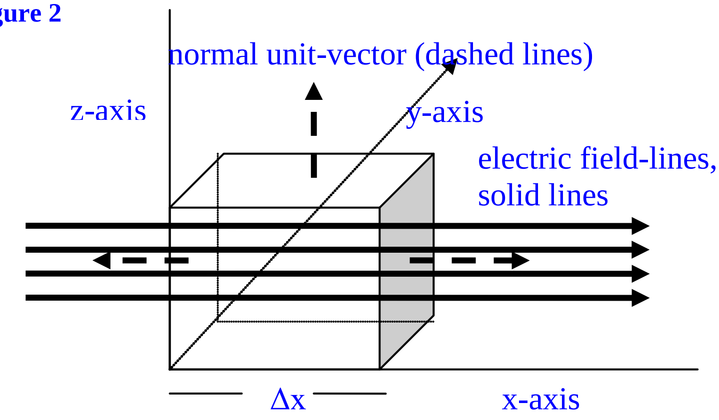
We are going to show that the total flux is always 0 for an electric field (even if it is not uniform) passing through a closed surface as long as no new fieldlines are being created inside of the closed surface, i.e. inside the volume within the surface.

So, let us figure out how the total flux changes if we allow the electric field to change from location to location within a small volume of sides Δx.
In our example the electric field enters the cube on the left as the x-component $E_x(x,y,z)$ and leaves the cube as $E_x(x+\Delta x,y,z)$.

24.2 Localized flux: divergence and Gauss' Law.

If $\vec{E}$ is not constant in the x direction and we use infinitesimal surfaces we get a negative entering infinitesimal flux at x and a positive leaving flux at x+Δx, like so

Figure 2



We call the infinitesimal flux through a closed infinitesimal surface $\Delta\Phi_{total} = \Delta\Phi_{in} + \Delta\Phi_{out}$.
We separate it into the sum of the ingoing flux and the outgoing flux. Note the obvious: A closed surface encloses a volume. The Δ symbol indicates that we are calculating a very small flux, which comes about by the small surface enclosing a small volume.

(24.6)

$\Delta\Phi_{in} = -E_x(x,y,z)\cdot\Delta y\Delta z$ (x-component of $\vec{E}$•infinitesimal surface $\perp$ to it)

$\Delta\Phi_{out} = E_x(x+\Delta x,y,z)\cdot\Delta y\Delta z$

total net "infinitesimal" flux $=\Delta\Phi_{total} = \Delta\Phi_{out} + \Delta\Phi_{in} = E_x(x+\Delta x,y,z)\cdot\Delta y\Delta z - E_x(x,y,z)\cdot\Delta y\Delta z$

$\Delta\Phi_{total} = \left[E_x(x+\Delta x,y,z) - E_x(x,y,z)\right]\Delta y\Delta z$

If we multiply this expression by $1=\frac{\Delta x}{\Delta x}$, we get the result that

$$\Delta\Phi_{total} = \left[E_x(x+\Delta x) - E_x(x)\right]\Delta y\Delta z\frac{\Delta x}{\Delta x} = \frac{E_x(x+\Delta x) - E_x(x)}{\Delta x}\Delta y\Delta z\Delta x$$

We recognize that the fraction is (in the limit of $\Delta x \to 0$) equal to the partial derivative of the x component of the vector field $\vec{E}$, namely $E_x(x,y,z)$:

(24.7) $$d\Phi_{total} = \lim_{\Delta V\to 0}\Delta\Phi_{total} = \lim_{\Delta x\to 0}\frac{E_x(x+\Delta x) - E_x(x)}{\Delta x}\Delta y\Delta z\Delta x = \frac{\partial E_x}{\partial x}dxdydz = \frac{\partial E_x}{\partial x}dV$$

This accounts only for the flux through the two sides perpendicular to the x-axis. If we compute the flux through the sides perpendicular to the y and z axes we get the total flux through the whole surface of the cubical volume.

The infinitesimal total or net localized flux through all six sides of a cube becomes :

(24.8) $$d\Phi_{total} = d\Phi_{out} + d\Phi_{in} = \underbrace{\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right)}_{div\vec{E}} dxdydz = \underbrace{\vec{\nabla} \cdot \vec{E}}_{div\vec{E}} dV$$

This looks like some kind of a derivative.

(24.9) $$\boxed{d\Phi_{total} = \underbrace{\vec{\nabla} \cdot \vec{E}}_{div\vec{E}} \cdot dV = div\vec{E} \cdot dV}$$

The change of flux divided by the change in volume through which the flux is defined is sometimes called a volume derivative and is called “the divergence of the vector-field passing through the volume”

(24.10) $$\boxed{\underbrace{\vec{\nabla} \cdot \vec{E}}_{div\vec{E}} = \frac{d\Phi_{total}}{dV}}$$

We come to the conclusion that the total **infinitesimal flux** of a vector field through a closed infinitesimal surface (surrounding the point x,y,z) is equal to the **divergence** of that vector field at the point x,y,z times the infinitesimal volume inside that surface**.**

From (24.8) we recall the definition of the div operator:

(24.11) $$\boxed{div\vec{E} = \vec{\nabla} \bullet \vec{E} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}}$$

Problem: Calculate the divergence of $\vec{E} = \left\langle 3xy, 2zy^2, \ln(z) \right\rangle$

To extend the equation (24.9) to the whole surface we must integrate on the left over the closed surface $\displaystyle\oint\!\!\!\oint_{closed\ surface} d\Phi = \oint\!\!\!\oint_{closed\ surface} \vec{E} \cdot d\vec{A}$

while at the same time integrating on the right side over the volume

$$\iiint_{\substack{volume\\ inside\\ the\ closed\\ surface\ A}} div\vec{E} \cdot dV$$

The generalization of this argument leads to the equality between the total flux of a vector field through a **closed surface** and the volume integral of the divergence of this vector field taken over the volume enclosed by that surface.

This mathematical theorem is called the **Gauss theorem**:

(24.12)
$$\Phi_E = \underset{\text{closed surface S}}{\oiint} \vec{E}\cdot d\vec{A} = \underset{\substack{\text{volume contained}\\ \text{inside the closed}\\ \text{surface S}}}{\iiint} div\vec{E}\cdot dV$$

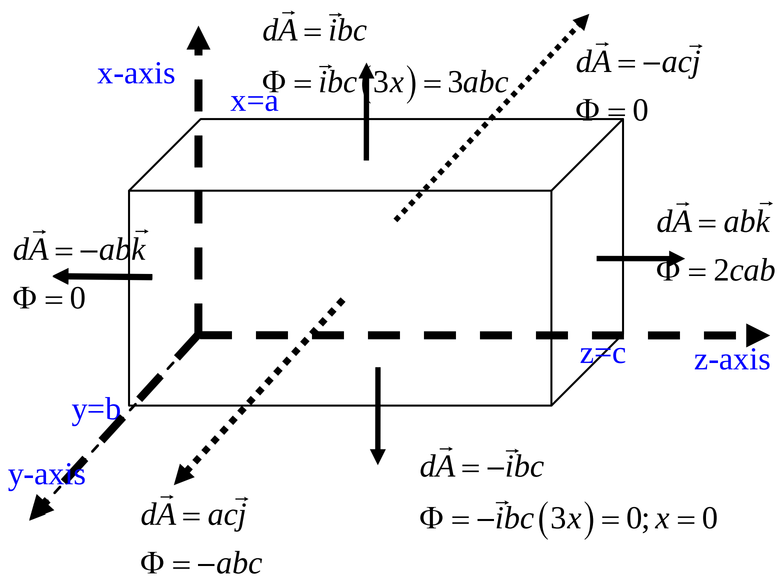
The theorem enables us to convert a **volume integral** into a **surface integral** (closed surface), and vice versa. **The surface is any surface enclosing the volume, because the integral on the right side only contributes if and where** $div\vec{E} \neq 0$.

In equation (24.6) we saw that the total flux of $\vec{E}$ through an infinitesimal surface surrounding an infinitesimal volume was equal to the change of that field inside of the infinitesimal volume.

(24.13)
$$\Delta\Phi = \Delta\Phi_{out} + \Delta\Phi_{in} \rightarrow d\Phi_{total} = \frac{\partial E_x}{\partial x} dxdydz = \frac{\partial E_x}{\partial x} dV$$

If no new charges are created inside of this infinitesimal volume, (no sinks and no sources), then the electric field does not change $\frac{\partial E_x}{\partial x} = 0$ and the total infinitesimal flux is 0. We can integrate this over any macroscopic volume, whose boundary is the Gaussian surface. This represents the Gaussian theorem. If $div\vec{E} = 0$ everywhere in the volume, then the total flux through the volume is 0.

Problem:
Calculate the flux of the vector field $\vec{E} = \langle 3x, -y, 2z \rangle$ through the surface enclosing the rectangular volume being positioned along the x, y, and z axis and having the dimensions



according to the figure:

We calculate the flux in two ways, one, by calculating the volume integral of the divergence of the vector field, which is equal to 4. Thus, the answer is 4abc.
Now we calculate the flux directly.

$$\Phi = \oiint_{\partial V} \langle 3x, -y, 2z \rangle \bullet d\vec{A}$$

(24.14) The surface vectors are perpendicular to the respective surfaces, point to the outside of the volume, and have the area as magnitude.

The surface integral of the left surface is equal to 0 because z=0 $\Phi_L = \iint_{xy} 2z\vec{k} \cdot \left(-\vec{k}\right) ab = 0$. The surface integral on the right side is equal to $2z\vec{k} \cdot \left(\vec{k}\right) ab = 2cab$ because z=c . You must take the scalar product between the vector field and the six surfaces of the volume. The surface integral at the bottom is 0 because x=0. The surface integral through the top is: $3x\vec{i} \cdot \vec{i} \cdot bc = 3abc$. The flux through the back, in the negative y-direction, is 0 because y=0. The surface integral through the front is $-y\vec{j} \cdot \vec{j}ac = -bac$
If we same up all the contributions through the six sides we get 4abc, as expected.

Problem: Show that the flux of the vectorfield <x,y,z> through a spherical shell of radius R is equal to $4\pi R^3$.

Optional: Illustration of flux in liquid flow:
In the context of the **steady state flow of a liquid** through a pipe we called this conservation of matter the **continuity equation**:
(24.15)

$$\rho = \text{mass density of the liquid} = \frac{m}{V}.\ \Delta m = \rho \Delta V = \rho A \Delta x \Rightarrow \frac{\Delta m}{\Delta t} = \rho \frac{\Delta V}{\Delta t} = \rho A \frac{\Delta x}{\Delta t}$$

$$A_{in} \mathrm{v}_{\mathrm{in}} - A_{out} \mathrm{v}_{\mathrm{out}} = 0;$$

$$\rho\left(-A_{in} \cdot \frac{dx_{in}}{dt} + A_{out} \frac{dx_{in}}{dt}\right) = \text{total flux of liquid through a pipe} = \oiint_{pipe} \vec{\mathrm{j}} \cdot d\vec{A}$$

$\vec{\mathrm{j}} = \rho \vec{\mathrm{v}}$ is the liquid density vector, which plays the role of the electric field vector. Both can change in magnitude and direction from point to point.

which describes the fact that the amount of liquid entering a closed surface is equal to the liquid leaving it. It is really just the conservation of mass. If Δm is the mass flowing through the closed surface (think of a volume as described by a cylindrical pipe) from time t to t+Δt, then the surface integral below is 0 and says, "**what flows in" equals to "what flows out**". If Δm is positive that means that more mass flows out of the pipe than flows in. This is possible if the density of the liquid inside the volume contained by the closed surface changes, or if there is a source (or sink) of liquid, (a sprinkler system, or a set of holes).

$$\oint\!\!\oint_{\vec{A}} \rho\vec{v}d\vec{A} = \lim_{\Delta t\to 0}\frac{\Delta m}{\Delta t};$$

(24.16) seen from the point of view of the volume the amount of M flows out of it in the time dt.

$$div\rho\vec{v}=-\frac{\partial\rho}{\partial t} \Rightarrow \oint\!\!\oint_{A} div\rho\vec{v}d\vec{A} = -\frac{dM}{dt}$$

If the amount of mass lost is being replenished at all times, we are dealing with steady state flow and we have $div\rho\vec{v}=0 \Rightarrow \oint\!\!\oint_{A} div\rho\vec{v}d\vec{A} = 0$.

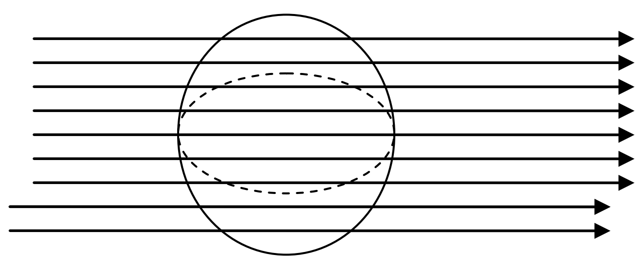
24.2 Gauss' Law for Electrostatics.

The electric field lines behave like the **stream lines of a fluid**, with the exception that every single charge creates an electric field line. Also, the electric field in the classic field model does not involve the movement of a substance. The field is static.

A positive charge acts as a source of electric field lines (a new field line emerges from it), a negative charge acts as a sink (a field-line sinks into it).

Figure 3

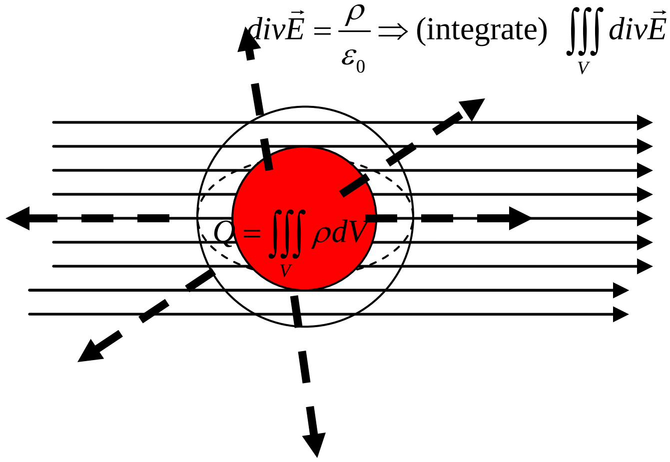
$$div\vec{E} = 0 \Rightarrow \Phi_E = 0$$



One way in which an electric field can change inside an infinitesimal volume is through the presence of **electric charges**, (which act as a source of electric fields), or, in a continous situation, a **charge density**. Instead of having single charges at various locations, we can have a charge density as a function of x, y, and z. In most of our cases this density is uniform. It is the same everywhere inside of a volume, along a surface or line. As we know from the beginning of these lectures, electric field lines emerge from positive charges and end in negative charges. At the location of any charge there is therefore a dramatic change in the electric field. (It is called a singularity.)

$$div\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \text{(integrate)} \iiint_V div\vec{E}\cdot dV = \oiint_A \vec{E}d\vec{A} = \Phi_E = \iiint_V \frac{\rho}{\varepsilon_0}dV = \frac{Q}{\varepsilon_0}$$

$$\oiint_A \vec{E}d\vec{A} = \frac{Q}{\varepsilon_0}$$



If there is such a charge density it turns out that the **local spatial change** of $\vec{E}$, the divergence of $\vec{E}$, is proportional to the charge density ρ. If there are no charges (sources of $\vec{E}$-lines) or sinks (end-points of $\vec{E}$-lines) inside of the volume, then $div\vec{E} = 0$, otherwise we have:

(24.17)

$$div\vec{E} = \frac{\rho}{\varepsilon_0} = \lim_{\Delta V \to 0} \frac{\Delta\Phi_{\vec{E}}}{\Delta V}$$

Where we also assume that the charge ρ is a constant.
In the most general case it could be dependent on x, y, z, t.

We see immediately that this is where the rule comes from that **<u>the number of field lines per unit area is proportional to the charge</u>**. It is simply the qualitative expression of the above law.

For a constant field perpendicular to a surface we get $\vec{E}\bullet\vec{A} = \frac{\rho V}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$

If we know the total charge in a volume or area we need to integrate both sides and apply Gauss law:

(24.18)

Gauss' law for electrostatics, or 1st Maxwell law:

$div\vec{E} = \frac{\rho}{\varepsilon_0}$, ρ is the positive or negative charge density measured in $\frac{\mathrm{C}}{\mathrm{m}^3}$.

$$\iiint_{volume} div\vec{E}\cdot dV = \iiint_{volume} \frac{\rho}{\varepsilon_0}dV = \frac{Q}{\varepsilon_0} = \oiint_{\substack{\text{any surface}\\ \text{enclosing}\\ \text{the volume}}} \vec{E}\bullet d\vec{A} = \Phi_E$$

<u>24.4 Consequences of Gauss' Law</u>:

From this we can verify:

- The electric flux through any closed surface is 0:

If there are no charges inside or if the net charges inside are 0. This does not mean however that there are no electric fields present inside of the closed surface. We will see later that electric fields can also be created in ways which do not depend on singular charges.

- The particular electric fields we study here do come from charges. This fact is entirely described by the **differential law** $div\vec{E} = \frac{\rho}{\varepsilon_0}$
- Or by the integral law (Gauss) $\oiint_A \vec{E}d\vec{A} = \frac{Q}{\varepsilon_0}$

If we are given a charge distribution with a high degree of symmetry and want to calculate the electric field at a given location, we try to **put a Gaussian surface through that location** and see if we can determine the flux through symmetry considerations. If we succeed, the flux equals the total net charge inside of the surface.

24.4a Analogy between the electric field and the gravitational field:

We have had another situation where a field originated in a source. That was the case of the gravitational field which started in any particle of mass or energy. The corresponding differential law would be for the gravitational field Γ which originates at any mass distribution.

(24.19)
$$div\Gamma = -4\pi G\rho_g$$
$G = 6.673 \cdot 10^{-11}$ (SI) universal gravitational constant;

ρ_g is the mass-density in $\frac{\text{kg}}{\text{m}^3}$.

The gravitational force on a mass m_1 was obtained by

(24.20)
$$\vec{F}_g = m_1\vec{\Gamma}$$

We see that the only difference in the two laws (24.19) and (24.18) lies in the value of the constant factors in front of the respective densities. The minus sign in the case of the gravitational field means that gravitation is always attractive.

We remember from the study of gravitation inside a sphere with uniform mass density that the field inside of the earth at a radius r from the center stems only from the mass inside a shell with that radius. Gauss' law explains easily why that must be the case. We also saw that inside a hollow spherical shell there is no gravitational field (no mass inside of a Gaussian surface, means no field originates there). Gauss' laws shows why. Where there is no mass, the divergence must be 0.

24.5 Electric field of a positively charged conducting shell:

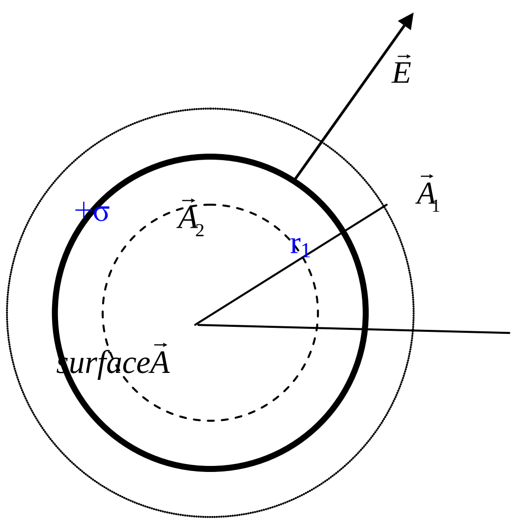
There are positive charges on the conducting surface A with uniform (constant) charge distribution σ. This means that in **electrostatic equilibrium** the electric field lines are perpendicular to the surface, pointing outward. The surfaces A_1 and A_2 are concentric to A. As the charges repel each other and are free to move they gather at the outside of the shell. There are no charges inside the shell.

$div\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow apply$ volume integral

with the volume being a sphere of radius r_1

We use the mathematical Gaussian law to convert the volume integral of $div\vec{E}$, to a surface integral of E. The surface is the surface of the volume with radius r_1.

The electric field has the same magnitude everywhere and is always perpendicular to the surface, which means it is everywhere parallel to the normal vector on the Gaussian surface also. Therefore, the surface integral is simply the constant field multiplied by the surface. The result is **Coulomb's law**.



$$\iiint_{\substack{\text{Gaussian sphere}\\ \text{with radius } r_1}} div\vec{E}\cdot dV = \iiint_{\substack{\text{Gaussian sphere}\\ \text{with radius } r_1}} \frac{\rho}{\varepsilon_0} dV = \frac{Q}{\varepsilon_0}$$

$$\downarrow$$

$$\iint_{\substack{\text{Gaussian surface}\\ \text{with radius } r_1}} \vec{E}\cdot d\vec{A}_1 = E4\pi r_1^2 \Rightarrow E = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_1^2} = k_e\frac{Q}{r_1^2}$$

(24.21) $$\boxed{E = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}}$$

This whole procedure is summarized in the Gaussian law: The net flux through any closed surface is equal to the total charge contained inside of that surface, divided by the permittivity constant**. It does not matter if you have a line, surface, or volume charge density. A volume integral will give a non 0 contribution only where there are charges. In the case of a surface charge density like above, the integral on the right side becomes a simple product** $\frac{\rho}{\varepsilon_0}V = \frac{\sigma}{\varepsilon_0}A = \frac{Q}{\varepsilon_0}$; Q is the net charge contained inside the volume

(24.22) $$\boxed{\Phi = \iint_{\substack{\text{Gaussian surface}\\ \text{with radius } r_1}} \vec{E}\cdot d\vec{A}_1 = \frac{Q}{\varepsilon_0}}$$

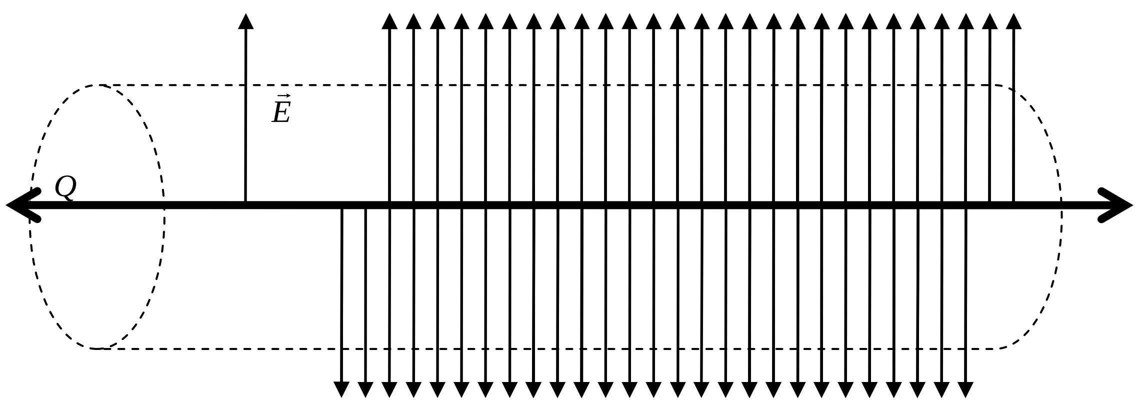
Note that you **<u>must not</u>** write $div\vec{E} = \frac{\sigma}{\varepsilon_0}$ since this is dimensionally **<u>incorrect</u>**.

24.6 Using Gauss's Law to Calculate the Electric Field:

Conductors in electric equilibrium : As free charges (electrons) placed on a conductor can move with only the slightest electric field difference, they will move until they are separated from each other as far as possible. Consequently, all free charges on a conductor in electric equilibrium must sit on top of the surface, and the electric field inside of the conductor is 0 (This is true for homogenous materials only, i.e. the same metal. **At junctions of different metals small electric fields do exist**.).
The charges on top of the surface are bound slightly to the conductor by the small internal atomic forces (electron shells). The electric fields are perpendicular to the surface. On sharp edges the density of field lines is greatest.

Problem: Find the electric fields inside and outside of two concentric conducting spheres. The inside sphere has a charge of 2Q. The outside sphere has a charge of –Q. Start by drawing a picture with the two concentric spheres and apply Gauss' law. We know that inside of either conductor the electric field is 0. The radius of the smaller sphere is a. The inner radius of the outer shell is b, the outer radius of the outer shell is c. The electric field between b and c is $+\frac{k_e 2Q}{r^2}$. The electric field outside of the outer shell is: $+\frac{k_e Q}{r^2}$

24.7 Electric field of an infinite line charge:
We know immediately that the electric field is perpendicular to the line, pointing outwards for a positve charge. The line being infinite, we can place a vertical axis on which to determine the electric field anywhere. The horizontal components of the electric fields created by the charges to the left and the right of this perpendicular axis cancel each other out. The resulting electric fields are therefore perpendicular to the line. They spread radially in all directions. The field is only dependent on the perpendicular distance r. At any constant distance r from the line the electric field must be the same. Thus, we choose a concentric cylinder of radius r and length L



as our Gaussian surface and calculate the total flux through it. We immediately get that

(24.23) $$\Phi = \underset{\substack{\text{Gaussian}\\\text{cylinder of}\\\text{radius r and}\\\text{length L}}}{\oiint} \vec{E}\cdot d\vec{A} = \underbrace{2\pi r L}_{\substack{\text{surface area}\\\text{of the mantel}\\\text{of the cylinder}}} \cdot E = \frac{\text{charge inside the cylinder}}{\varepsilon_0} = \frac{Q}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0}$$

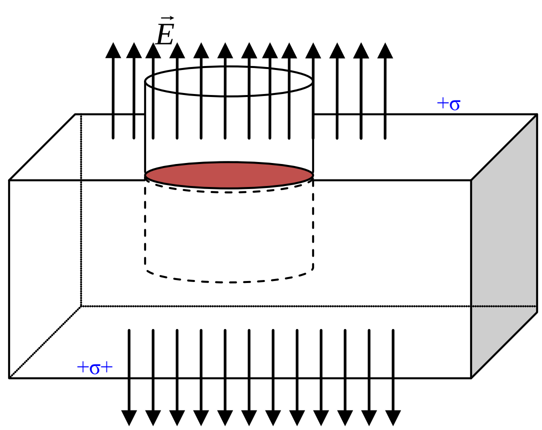
The flux through the top and bottom of the Gaussian cylinder is 0. (Why?).
We therefore get the electric field as:

(24.24) $$\boxed{\vec{E}(r) = \frac{\lambda \vec{u}_r}{\varepsilon_0 2\pi r} = \frac{2k_{el}\lambda}{r}\vec{u}_r}$$

The electric field is perpendicular to the wire and decreases with 1/r with the distance r from the wire.

24.8 Electric Field of a Thick Slap of Conducting Material.

Find the electric field created by a thick and infinitely large slap of **conducting material** with a surplus positive charge on it with charge density σ**.** Note that in electrostatic equilibrium all charges are uniformly distributed on the outside of the conductor. **The electric field inside of the conductor, between the charged surfaces is 0**.
The charge contained within the Gaussian cylinder is σA. We argue like in the case of the infinitely long line in the preceding example to see that the electric field lines must be perpendicular to the surfaces. We place a Gaussian cylinder through the slap of material with the circular ends parallel to the surfaces. The upper end of the cylinder has area A and intercepts a point at a distance r from the charged surface. We place the lower circular end inside the slap, where we know that there is no electric field.



Note that at the origin of each arrow there is a charge.

The surface integral equals
$$\iint_A \vec{E}\cdot d\vec{A} = \oiint_{\partial V} \vec{E}\cdot d\vec{A} = E\cdot A.$$
Therefore we have:

(24.25)
$$\iiint_{\text{Gaussian cylinder}} div\vec{E}\, dV = \iiint_{\text{Gaussian cylinder}} \frac{\rho}{\varepsilon_0} dV = \frac{\sigma A}{\varepsilon_0}; A = \pi r_1^2$$
$\downarrow$
$$\oiint_{\substack{\text{Gaussian}\\\text{closed surface}}} \vec{E}\cdot d\vec{A} = E\pi r_1^2 = EA \Rightarrow E = \frac{\sigma}{\varepsilon_0}$$

The volume integral on the right is $\iiint_{\text{Gaussian cylinder}} \frac{\rho}{\varepsilon_0} dV$. As all charges are sitting on top of the surface (above or below), the contribution to the integral come strictly from the surface, therefore the result of the integration is $\frac{\sigma A}{\varepsilon_0}$. The electric field inside the conductor is therefore 0.

Note that in our setup there is only a flux through the horizontal upper and lower surface of the cylinder. The electric field inside of the conducting slap is 0. Thus the electric field **outside of a charged plane conductor** is

(24.26) $$\boxed{E = \frac{\sigma}{\varepsilon_0}}$$

If we use a Gaussian cylinder that passes through the slap, we get the same field on the other side, same amount of charge, same surface.

We could also take the outside of a box as our Gaussian surface (instead of the cylinder). We place the box symmetrically around a segment of a finite slap of charge. Inside of the the slap there is no electric field, therefore no flux. Similarly, the flux is 0 at the sides where the electric field is perpendicular to the surface normal vectors. Therefore, the only contribution to the flux comes from the top or the bottom surface,

(24.27)
$$\iiint\limits_{\text{Gaussian box}} div\vec{E}dV = \iiint\limits_{\text{Gaussian box}} \frac{\rho}{\varepsilon_0}dV = \frac{\sigma A}{\varepsilon_0}$$
$$\downarrow$$
$$\oiint\limits_{\substack{\text{Gaussian}\\ \text{closed surface}}} \vec{E}\cdot d\vec{A} = EA = \frac{\sigma A}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{\varepsilon_0}$$

Note that the electric field generated by a conducting sheet is twice as large as that of a sheet of charge, because all field lines originating from charges go only to one side of the sheet.

24.9 Electric Field Created by a Sheet of Charges:

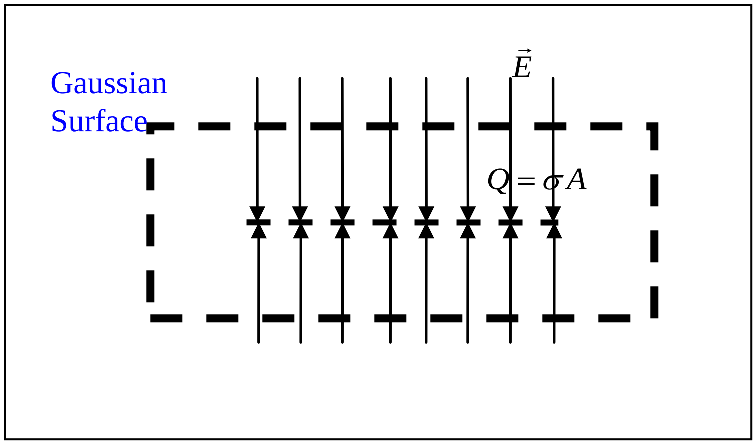
We get**:**

(24.28) $$-2EA = \iiint_V \frac{\rho}{\varepsilon_0}dV = \frac{\sigma A}{\varepsilon_0}$$

A **sheet of charges** has electric fields emanating to **both sides** of the **single sheet of charges.** The electric field generated by negative charges points in the opposite direction of the normal vectors to the surface, thus the negative sign. The surface is 2A (top and bottom of the Gaussian box), whereas the surface of charges is A. The surface integral contributes only in the area on top of the charges. For a sheet of electrons we get:

(24.29) $$div\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \Phi = \oiint\limits_{box\text{ surrounding sheet}} \vec{E}\cdot d\vec{A} = -EA - EA = \frac{Q}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

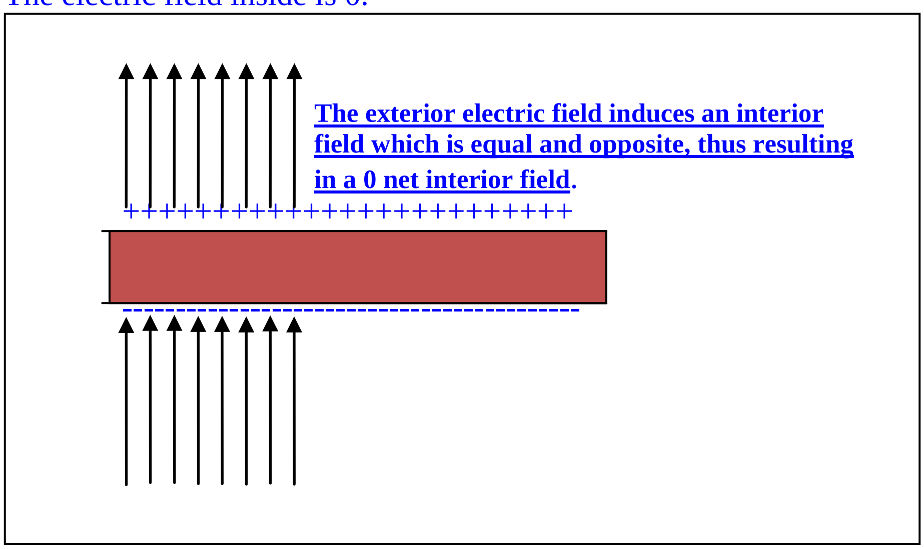
(24.30) $$\boxed{E = -\frac{\sigma}{2\varepsilon_0}}$$



24.10 Conductor in Electrostatic Equilibrium Inside of an Electric Field :

If we place an electrically neutral slap of conducting material into a preexisting uniform electric field $\vec{E}_{ext}$ the field will move the **negative electrons** to the lower side of the slap, which results in having **positive charges** at the top of the slap. Thus, an interior electric field will be established until it cancels the exterior electric field. As long as a field imbalance exists, electrons will move lower. This will only stop when electrostatic equilibrium has been established, i.e. when the interior electric field inside the slap is 0.

- The electric field inside is 0.



- All charges are distributed on the outside.
- The electric field just outside the charged conductor is $E = \frac{\sigma}{\varepsilon_0}; \sigma = \frac{Q}{A}$
- On an irregularly shaped conductor, σ is greatest where the radius of curvature is smallest. Sharp metallic points have the stronger electric fields. We can visualize a sharp corner in a metall as a tiny semi-sphere which creates an electric field outside of it

proportional to $1/r^2$. A small radius will lead to a strong electric field, i.e. relatively dense electric field lines, diverging from the sphere. This density is proportional to the surface charge density.

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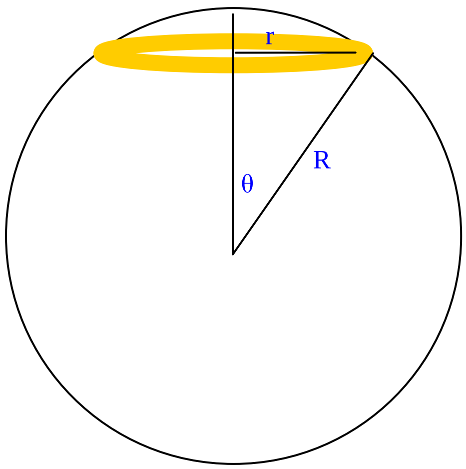
24.11 Flux and Solid Angle:

Calculate the circular surface element of a spherical shell with radius R and angle θ. The circular surface element has a horizontal radius r=Rsinθ.



(24.31)

$ds = Rd\theta$ tangential line element of the circular cross-section of the sphere. dA is the circular strip of circumference 2πr with height ds=Rdθ

$$\iint dA = \underbrace{\int 2\pi r \cdot ds}_{\text{surface area}} = \int_0^\theta 2\pi r \cdot Rd\theta = 2\pi R \int_0^\theta \underbrace{\sin\theta \cdot R}_{r} d\theta$$

$$= 2\pi R^2 \int_0^\theta \sin\theta d\theta = 2\pi R^2 \left(1 - \cos\theta\right)$$

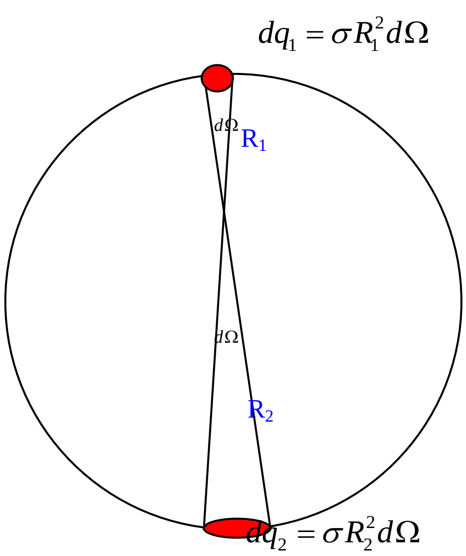
For θ=π/2 we get the surface area of the upper hemisphere, which is $2\pi R^2$. **The stereo angle which subtends the area in (24.31) is called the solid angle Ω. We can write the area then as Ω R^2**.

The electric field created by a centrally located charge q inside this sphere of radius R is radial. It generates a flux through the above surface of:

(24.32) $$\iint_{\substack{spherical \\ surface}} \vec{E}d\vec{A} = \int_0^\theta \frac{k_e q}{R^2} 2\pi R^2 \sin\theta d\theta = 2\pi k_e q \int_0^\theta \sin\theta d\theta = 2\pi k_e q\left(1 - \cos\theta\right)$$

This means the flux is independent of R.

The concept of the **solid angle** allows one to also quickly see why there is no electric or gravitational field inside the sphere, without using Gauss's theorem. The forces from opposite sides at any point inside the sphere cancel each other out **because the solid angle is the same**. The charge on opposite sides of the point in question is proportional to the areas which are equal to ΩR_1^2 and ΩR_2^2 respectively. There are only vertical forces in the direction of the bisector of the angle remaining. These forces are equal to, respectively,



$$\frac{k_e dq_1}{R_1^2} = \frac{k_e \sigma d\Omega \cdot R_1^2}{R_1^2} = -k_e \sigma \cdot d\Omega$$

and $$\frac{k_e \cdot dq_2}{R_2^2} = \frac{k_e \sigma d\Omega \cdot R_2^2}{R_2^2} = +k_e \sigma \cdot d\Omega$$

If we make the angle of infinitesimal size the areas dA are perpendicular to the radius. The radius cancels out and, as the forces from the top point to the opposite direction as the forces from the bottom, the net force at the intersecting point is 0.