Problems: See website

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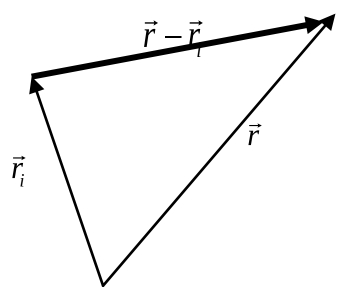
Notation:

$$\vec{r} = \underbrace{\langle x, y, z \rangle}_{brackets} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r} - \vec{r}_i = \langle x - x_i, y - y_i, z - z_i \rangle$$

$$\vec{u}_r = \text{unit vector} = \frac{\vec{r}}{r} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{r} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}$$

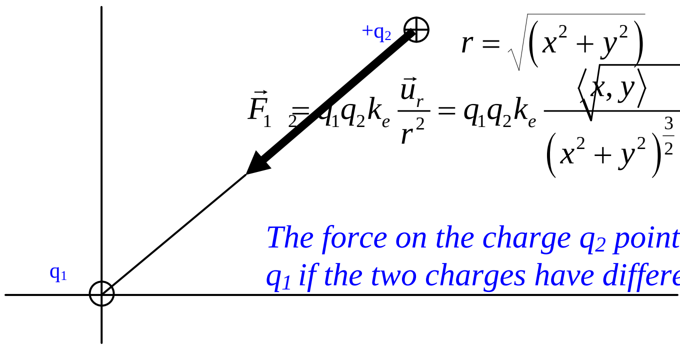
$$\vec{F}(x, y, x) = F_x(x, y, x)\vec{i} + F_y(x, y, x)\vec{j} + F_z(x, y, x)\vec{k}$$



23.1 Electric Fields and Forces:

An electric charge is a fundamental property of matter, similar to the property of mass. Electric charge comes in two flavors, one is called negative charge, the other is called positive charge. Electric charge is designated with the letter q, preceded by a + or -. The charge on the elementary **fundamental particle** electron is negative, the charge on a proton, which is also a fundamental particle, is positive. Charge, being a fundamental property, cannot be further analyzed. Unlike mass, charge comes in discrete multiples of one single charge quantum which has the value of

Figure 1



$$r = \sqrt{(x^2+y^2)}$$

$$\vec{F}_{1\,2} = q_1 q_2 k_e \frac{\vec{u}_r}{r^2} = q_1 q_2 k_e \frac{\langle x, y\rangle}{(x^2+y^2)^{\frac{3}{2}}} = \frac{q_1 q_2 k_e}{r^2}\frac{\langle x, y\rangle}{\sqrt{x^2+y^2}}$$

The force on the charge q_2 points to the charge q_1 if the two charges have different signs.

$e^- = -1.602\ 176\ 462\ (63)\cdot 10^{-19}$ Coulombs in the SI system.

A typical macroscopic charge is in the order of a few micro-Coulombs or µC. Any such charge consists of an integer number of elementary charges. $Q=Ne^-$

As we have learned earlier in the context of atoms, **<u>unlike</u> charges attract each other**, **<u>like</u> charges repel each other** according to a law which is very similar to Newton's law of universal gravitation. On a macroscopic level we are only dealing with electrons: a surplus of electrons results in a negative total charge; a deficiency of electrons results in a positive charge.

All molecules and atoms are electrically neutral if observed from a distance because, obviously, the number of electrons (negative charges in the atomic shells) is balanced by an equal number of protons (positive charges at the nuclei).

23.2 Electric Induction.

Materials can be classified according to the ease at which charges can move inside of them. We distinguish between **conductors,** in which electrons can move relatively freely and **insulators** in which electrons are bound to molecules. Electric conductors are typically metals. In copper for example, the electrons on the outmost shell of the atoms are shared by the all copper atoms in a sample. Only a slight exterior force is necessary to make all these electrons move in one direction or another. Such a force is provided, for example, by excess charges on another piece of material. Those excess charges create an electric field around them, which is very much like a gravitational field, except that it only acts on other charges, and not on mass. The process in which these fields act on these other charges through empty space is called **induction**. Note that electrical insulators are also good **thermal insulators**, and electrical conductors are also good **heat conductors**. Insulators can be **locally charged**, whereas in a conductor, electrons always move around on the surface of the conductor until electric equilibrium is established and a **uniform distribution of charge** has been achieved. Note that a negative electric charge in a material means that there is a **surplus of negative charges** (electrons), whereas a positive charge means that there is a deficiency of negative charges (electrons.) **There are no free protons floating around in matter.**

23.3 Coulomb's Law.

The fundamental **force between two point charges q_1 and q_2** is given by **Coulomb's law**:
(The coordinate system has its origin in the charge q_1;
q_2 is located at the point P(x,y))

Coulomb's law of electrostatic attraction and repulsion:

Force of charge q_1 (located in (0,0)) on charge q_2(located in (x,y) = $\vec{F}_{21}$

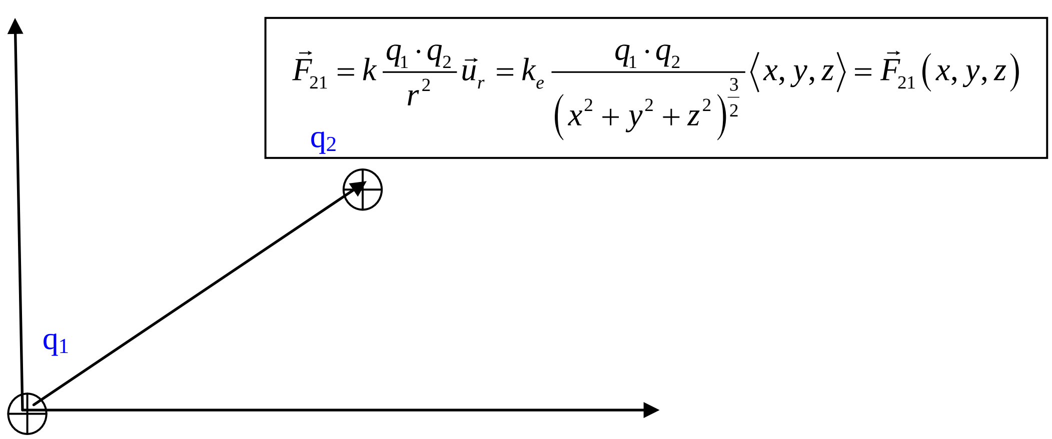
$$\vec{F}_{21} = k\frac{q_1 \cdot q_2}{r^2}\vec{u}_r = k_e \frac{q_1 \cdot q_2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\langle x,y,z\rangle = \vec{F}_{21}\left(x,y,z\right) \quad (23.1)$$

The force is attractive if $q_1 \cdot q_2 < 0$ the two charges have opposite signs;
it is a repellent force if $q_1 \cdot q_2 > 0$ both charges are alike, either positive or negative.

Coulomb's constant: $k_e = 8.9876 \cdot 10^9 \frac{Nm^2}{C^2} \approx 9 \cdot 10^9 \frac{Nm^2}{C^2}$

In the above definition we placed the charge q_1 into the origin of our coordinate system. This placement should remind you of our choice in gravitation theory where we placed the attracting body (sun or earth, usually) into the center of the coordinate system.

Figure 2



$$\vec{F}_{21} = k\frac{q_1 \cdot q_2}{r^2}\vec{u}_r = k_e \frac{q_1 \cdot q_2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\langle x,y,z\rangle = \vec{F}_{21}\left(x,y,z\right)$$

If we do not place the charge q_1 into the origin of the coordinate system we need to find the vector force with magnitude and direction, acting on charge q_2.
This force is parallel to the line connecting the two charges. It points away from q_1 if the force is repellent, it points towards q_1 if the force is attractive. Let us use the convention in which we write

(23.2)

$$\text{force on } q_2 \text{ created by } q_1: \vec{F}_{21} = k_e \frac{q_2 \cdot q_1}{(x_2-x_1)^2+(y_2-y_1)^2} \frac{\langle x_2-x_1, y_2-y_1\rangle}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} = k_e \frac{q_1 q_2}{\vec{r}_{21}^{\,2}} \vec{u}_{21}$$

$$\vec{F}_{21} = k_e \frac{q_1 \cdot q_2 \langle x_2-x_1, y_2-y_1\rangle}{\left[(x_2-x_1)^2+(y_2-y_1)^2\right]^{\frac{3}{2}}};$$

$$\vec{r}_{21} = \vec{r}_2 - \vec{r}_1 = \langle x_2-x_1, y_2-y_1\rangle, \vec{u}_{21} \equiv \frac{\langle x_2-x_1, y_2-y_1\rangle}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} = \frac{\vec{r}_{21}}{|\vec{r}_{21}|} = \frac{\vec{r}_{21}}{|\vec{r}_2-\vec{r}_1|}$$

The force on charge 2 created by a series of charges denoted q_i, with i=1,3,4,5,6…is given by the sum:

$$\vec{F}_2 = q_2 k_e \sum_i \frac{q_i \vec{u}_{2i}}{(\vec{r}_2-\vec{r}_i)^2} = q_2 k_e \sum_i \frac{q_i \langle x_2-x_i, y_2-y_i\rangle}{\left[(x_2-x_i)^2+(y_2-y_i)^2\right]^{\frac{3}{2}}}$$

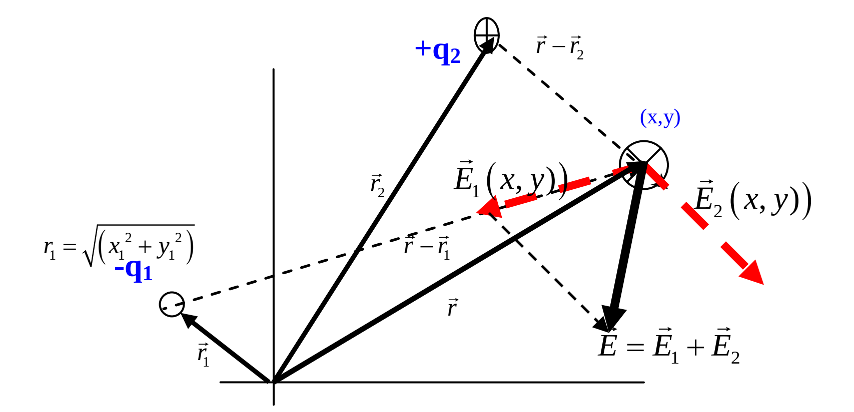
23.4 Definition of the electric field vector function.

Both charges create what is called an **electric field** around them. This field becomes observable if we place a third charge q_0 at any point in space. We designate that point with the vector $\vec{r} = \langle x, y, z\rangle$. (In the preceding section we calculated the force at this point, where the charge q_2 was located. The vector field $\vec{E}(\vec{r})$ at this point is defined as the resultant force acting on a positive test charge q_0 at this point divided by the positive unit charge q_0.

(23.3) $$\vec{E}_2 \equiv \vec{E}(x,y) = \frac{\vec{F}_2}{q_2} = k_e \sum_i \frac{q_i \vec{u}_{2i}}{(\vec{r}-\vec{r}_i)^2} = k_e \sum_i \frac{q_i \langle x-x_i, y-y_i\rangle}{\left[(x-x_i)^2+(y-y_i)^2\right]^{\frac{3}{2}}}$$

The value of the test charge is of course irrelevant as it is cancelled out.

Figure 3



(23.4)

$$\vec{E}(x,y,z) = \frac{\vec{F}(x,y,z)}{q_0}$$

The effect of this definition is that a negative charge $-q_1$ at point P_1 creates an attractive field at the location P(x,y,z) (with respect to the charge q_1),

(23.5) $$\vec{E}_1 = -q_1 k_e \frac{(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3} = -q_1 k_e \frac{(\vec{r}-\vec{r}_1)}{\left[(x_2-x_1)^2+(y_2-y_1)^2\right]^{\frac{3}{2}}}$$

whereas a positive charge q_2 creates a repellent field at P(x,y).

(23.6) $$\vec{E}_2 = +q_2 k_e \frac{(\vec{r}-\vec{r}_2)}{|\vec{r}-\vec{r}_2|^3}$$

The resultant electric field is the vector sum of the individual fields.

(23.7) $$\vec{E} = \vec{E}_1 + \vec{E}_2$$

For n charges located at the points (x_i, y_i) we get the resultant electric field at any point (x, y) in the plane by:

(23.8) $$\vec{E}(x,y) = k_e \sum_i \frac{q_i \vec{u}_{2i}}{(\vec{r}-\vec{r}_i)^2} = k_e \sum_i \frac{q_i \langle x-x_i, y-y_i \rangle}{\left[(x-x_i)^2+(y-y_i)^2\right]^{\frac{3}{2}}}$$

Conversely, the force acting on a charge q_3 placed in an electric field **E** is given by:

(23.9)

$$\vec{F}(x,y,z) = q_3 \vec{E}(x,y,z)$$

Note that both $\vec{E}$ and $\vec{F}$ are field vectors with components, in the case of $\vec{F}$: $F_{x,} F_{y,} F_z$, each of which is a scalar function of the variables x, y, z.

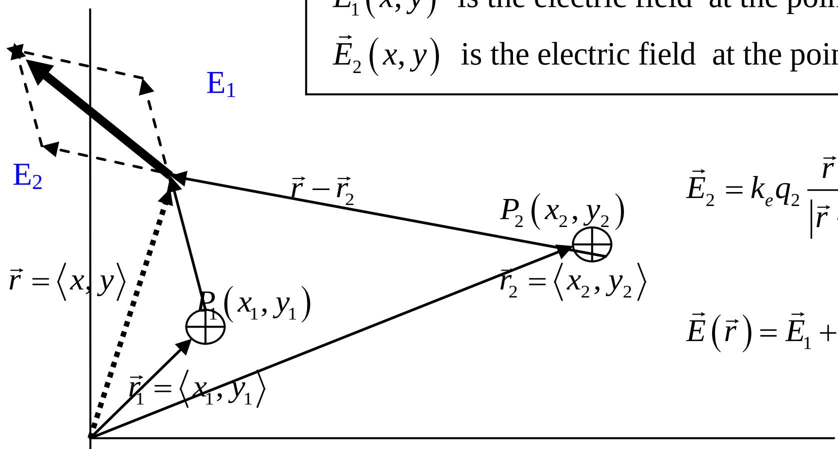
$$\vec{F} = \langle F_{x,} F_{y,} F_z \rangle = F_x(x,y,z)\vec{i} + F_y(x,y,z)\vec{j} + F_z(x,y,z)\vec{k}$$

Here is another example:

Figure 4

$\vec{E}_1(x,y)$ is the electric field at the point P(x,y) created by the charge q_1

$\vec{E}_2(x,y)$ is the electric field at the point P(x,y) created by the charge q_2



$$\vec{E}_2 = k_e q_2 \frac{\vec{r}-\vec{r}_2}{\left|\vec{r}-\vec{r}_2\right|^3}; \vec{E}_1 = k_e q_1 \frac{\vec{r}-\vec{r}_1}{\left|\vec{r}-\vec{r}_1\right|^3}$$

$$\vec{E}(\vec{r}) = \vec{E}_1 + \vec{E}_2 = k_e \sum_{i=1}^{2} q_i \left[\frac{\langle x-x_i, y-y_i\rangle}{\left((x-x_i)^2+(y-y_i)^2\right)^{\frac{3}{2}}} \right]$$

A word about **vector notation**:

$$\vec{r}-\vec{r}_1 = \langle x-x_1, y-y_1, z-z_1\rangle \text{ and } \left|\vec{r}-\vec{r}_1\right| = \sqrt{(\vec{r}-\vec{r}_1)^2} = \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}$$

(23.10)

$$\left|\vec{r}-\vec{r}_1\right|^3 = \left[(x-x_1)^2+(y-y_1)^2+(z-z_1)^2\right]^{\frac{3}{2}}$$

The total **resultant electric field** at the point $P(x,y,z)$ created by n charges q_i situated at the locations $\vec{r}_i = \langle x_i, y_i, z_i\rangle$ is the vector sum of these fields:

$$\vec{E}_{total} \text{at the point P(x,y,z)} = \vec{E}(\vec{r}) = \vec{E}(x,y,z) = k_e \sum_{i=1}^{n} \frac{q_i(\vec{r}-\vec{r}_i)}{\left|\vec{r}-\vec{r}_i\right|^3}$$

(23.11)

$$= k_e \sum_{i=1}^{n} \frac{q_i}{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2} \cdot \underbrace{\frac{\langle x-x_i, y-y_i, z-z_i\rangle}{\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}}}_{\text{unit vector pointing from charge } q_i \text{ to the point } \vec{r}} =$$

$$\vec{E}_{total}(x,y,z) = k_e \sum_{i=1}^{n} \frac{q_i \langle x-x_i, y-y_i, z-z_i\rangle}{\left[(x-x_i)^2+(y-y_i)^2+(z-z_i)^2\right]^{\frac{3}{2}}}$$

$\vec{u}_{ri}$ is the unit vector pointing from the charge q_i (+ or -) to the point P(x,y,z) where we calculate the electric field.

$$\vec{u}_{ri} = \frac{\vec{r}-\vec{r}_i}{\left|\vec{r}-\vec{r}_i\right|} = \frac{\left\langle x-x_i, y-y_i \right\rangle}{\sqrt{\left(x-x_i\right)^2+\left(y-y_i\right)^2}}$$

This direction is modified by the sign of the charge.

If the charge q_i is negative, the unit $\vec{u}_{ri}$ vector points away from P and towards the charge q_i.

Note the convention I use for designating vector components: They are enclosed in angle brackets <a,b>:(23.12)

location vector: $\vec{r}-\vec{r}_i \equiv \left\langle x-x_i, y-y_i, z-z_i \right\rangle$;

electric field vector: $\vec{E}(x,y,z) \equiv \left\langle E_x(x,y,z), E_y(x,y,z), E_z(x,y,z) \right\rangle \equiv \left\langle E_x, E_y, E_z \right\rangle$

In the earlier chapters about gravitation we also talked about the **gravitational field**. That concept is the same as the concept of the electric field. The gravitational field, which we may call $\vec{\Gamma}$ (capital γ, Greek for G.) is the result of a distribution of masses. We don't want to use G, because that can be confused with the universal gravitational constant G. The major difference is that **gravitational forces are always attractive**. There are no negative masses

(23.13) $$\vec{\Gamma}(x,y,z) = \vec{\Gamma}(\vec{r}) = -G\sum_{i=1}^{n} \frac{m_i\left(\vec{r}-\vec{r}_i\right)}{\left|\vec{r}-\vec{r}_i\right|^3} = -G\sum_{i=1}^{n} \frac{m_i\left(\vec{r}-\vec{r}_i\right)}{\left(\vec{r}-\vec{r}_i\right)^{\frac{3}{2}}}$$

This resultant gravitational field can be detected anywhere in space by placing a test mass there. In contrast to the electric fields all individual gravitational fields have a negative sign in front of the vectors because they are always attractive fields.

23.4a Electric field of a dipole: Let us calculate the resultant electric field in the case of a dipole, which consists of two equal and opposite charges separated by the distance 2a and at a large distance perpendicular to the connecting line of the two charges.

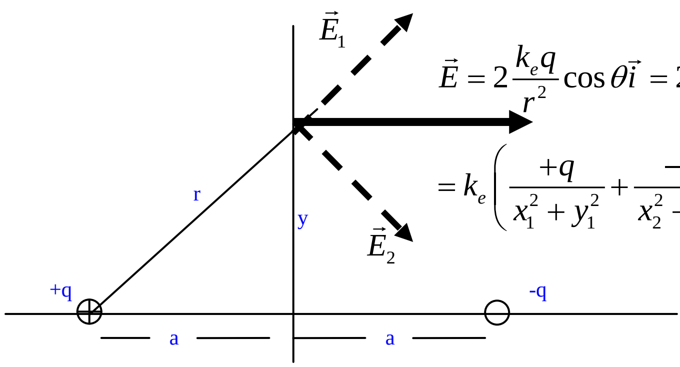
We place the two charges on the x-axis, symmetrically around the y-axis. This means that +q is located at x_1=-a, and –q is located at x_2=+a. (This configuration is called a **dipole**.) I use two approaches to find the result.
First, we simply calculate the resultant field **magnitudes** at the distance y from the origin. The **magnitude** of both fields is:

(23.14) $$E_1 = E_2 = \frac{kq}{r^2}; r^2 = a^2 + y^2$$

From a drawing we can see that the resultant field points in the positive x direction. The component is $2E_1\cos\theta$ with $\cos\theta = a/r$.
Therefore

$$r = \left(y^2 + a^2\right)^{\frac{1}{2}}$$

(23.15)
$$E = \frac{2kq}{r^2}\frac{a}{r} = \frac{2kaq}{r^3} = \frac{2kaq}{(y^2+a^2)^{\frac{3}{2}}} \to \frac{2kaq}{y^3} \text{ for y} \gg \text{a}$$



$$\vec{E} = 2\frac{k_e q}{r^2}\cos\theta\vec{i} = 2\frac{k_e q}{a^2+y^2}\frac{a}{\sqrt{a^2+y^2}}\vec{i} =$$

$$= k_e\left(\frac{+q}{x_1^2+y_1^2} + \frac{-q}{x_2^2+y_2^2}\right)\cos\theta\vec{i}$$

$$\vec{E}(0,y) = \frac{2akq}{\left(a^2+y^2\right)^{\frac{3}{2}}}\vec{i}$$

In the next approach we simply substitute the coordinates of the two charges, into (23.11), noting that the electric field is located at x=0 and y; $q_1 = +q; x_1 = -a; y_1 = y_2 = 0; q_2 = -q; x_2 = a$.

$$\vec{E}(x,y) = k_e\left(\frac{+q}{\left(x-x_1\right)^2+\left(y-y_1\right)^2}\frac{\left\langle x-x_1, y-y_1\right\rangle}{\left(\left(x-x_1\right)^2+\left(y-y_1\right)^2\right)^{\frac{1}{2}}} + \frac{-q}{\left(x-x_2\right)^2+\left(y-y_2\right)^2}\frac{\left\langle x-x_1, y-y_1\right\rangle}{\left(\left(x-x_2\right)^2+\left(y-y_2\right)^2\right)^{\frac{1}{2}}}\right) =$$

(23.16)
$$\vec{E}(0,y) = k_e\left(\frac{+q}{\left(0-x_1\right)^2+\left(y-y_1\right)^2}\frac{\left\langle 0-x_1, y-y_1\right\rangle}{\left(\left(0-x_1\right)^2+\left(y-y_1\right)^2\right)^{\frac{1}{2}}} + \frac{-q}{\left(0-x_2\right)^2+\left(y-y_2\right)^2}\frac{\left\langle 0-x_2, y-y_2\right\rangle}{\left(\left(0-x_2\right)^2+\left(y-y_2\right)^2\right)^{\frac{1}{2}}}\right)$$

Now we use the fact that $x_1 = -a$ and $x_2 = +a; y_1 = y_2 = 0$ The y-components cancel and what remains is a vector field pointing in the +x direction.

(23.17)
$$\vec{E}(0,y) = k_e\left(\frac{+q}{a^2+y^2}\frac{\left\langle a, y\right\rangle}{\left(a^2+y^2\right)^{\frac{1}{2}}} + \frac{-q}{a^2+y^2}\frac{\left\langle -a, y\right\rangle}{\left(a^2+y^2\right)^{\frac{1}{2}}}\right) =$$

$$\vec{E}(0,y) = \frac{k_e q}{\left(a^2+y^2\right)^{\frac{3}{2}}}\left[\left\langle a, y\right\rangle + \left\langle a, -y\right\rangle\right] = \frac{2a k_e q}{\left(a^2+y^2\right)^{\frac{3}{2}}}\vec{i}$$

23.5 Electric field of a continuous distribution of charges:

For a continuous charge distribution the summation over individual charges turns into an **integral over the charge distribution**. We distinguish between **charge densities** for volume, surface, and line distribution of charges. I give the variables the index 1, to remind us that these variables are integration variables of definite integrals.

1. (23.188)

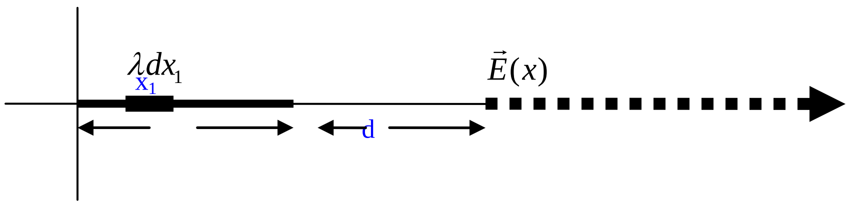
$$\rho \equiv \frac{Q}{V}; dq = \rho dV = \rho dx_1 dy_1 dz_1$$
$$\sigma \equiv \frac{Q}{A}; dq = \sigma dA = \sigma dx_1 dy_1$$
$$\lambda \equiv \frac{Q}{L}; dq = \lambda dx_1$$

The general formula for the electric field is then:
(23.19)

$$\vec{E} \text{ at a point } (x,y,z) = k_e \int \frac{dq_1}{r^2}\vec{u}_r = k_e \int \frac{\rho_1 dV_1}{r^2}\vec{u}_r =$$
$$\vec{E} \text{ at a point } (x,y) = k_e \int \frac{\langle x-x_1, y-y_1 \rangle}{\left[(x-x_1)^2 + (y-y_1)^2\right]^{\frac{3}{2}}} \rho_1 dV_1$$

The integral must be taken over the domain where the charges are being distributed. r is the distance from the charge element ρdV to the test point P.

23.5a Example 1 (rod of length l): Calculate the field created by a uniformly charged **rod of length l** with charge density λ. The point where you want to calculate the field is located at a distance d from the end of the rod along the axis of the rod. The charge is distributed over the rod of length l. It is over this length that the integration has to be performed.



Generally, you have to find the functional relationship between the infinitesimal charge element λdx_i and r_i, the distance from this element to the point where you need to calculate the electric field. Determine the direction of the field first, and then its magnitude.
Obviously, the direction of the resultant field is the positive x-direction. The magnitude of the field is given by the integral: (23.20)

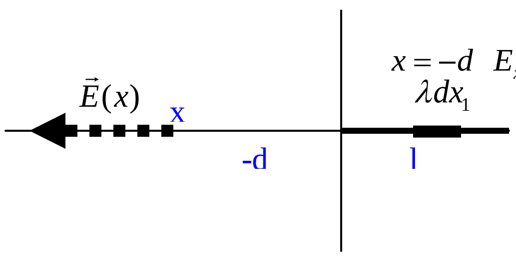
$$E_x(x) = k_e \int_0^l \frac{\lambda dx_1 (x - x_1)}{\left[(x-x_1)^2 + (y-y_1)^2\right]^{\frac{3}{2}}} = k_e \int_0^l \frac{\lambda dx_1 (x-x_1)}{\left[(x-x_1)^2 + 0\right]^{\frac{3}{2}}} = k_e \int_0^l \frac{\lambda dx_1}{(x-x_1)^2} =$$

$$= [\text{with } x - x_1 = z] - k_e \lambda \int \frac{dz}{z^2} = -k_e \lambda \left(\frac{-1}{z}\right) = k_e \lambda \left(\frac{1}{x - x_1}\right)\Bigg|_{x_1=0}^{x_1=l} = k_e \lambda \left(\frac{1}{x-l} - \frac{1}{x-0}\right) =$$

$$k_e \lambda \left(\frac{x - x + l}{(x-l)x}\right) = \frac{k_e l \lambda}{x(x-l)} = \frac{k_e Q}{x(x-l)}; with\ \lambda = \frac{Q}{l}$$

$$x = d + l \Rightarrow E_x = \frac{k_e Q}{(d+l)(d+l-l)} = \frac{k_e Q}{(d+l)d}$$

Note that x_1 is not the coordinate of the point P(x) where we calculate the electric field but the coordinate of the charge element λdx_1 to P. This distance varies from 0 to l, which are the limits of integration.
If we want to calculate the field at the distance d to the left of the charged line, the field will point to the left. It's magnitude will be the same.



$$x = -d \quad E_x = -\frac{k_e Q}{-d(-d-l)} = -\frac{k_e Q}{d(d+l)}$$

If we want to calculate the electric field at an arbitrary point in the x-y plane, we proceed as follows:
(23.21)

$$E_x(x,y) = k_e \int_0^l \frac{\lambda dx_1 (x-x_1)}{\left[(x-x_1)^2 + y^2\right]^{\frac{3}{2}}} = (\text{with z=x-x}_1\text{; dz=-dx}_1) = -k_e \lambda \int_{x_1=0}^{x_1=l} \frac{zdz}{\left[z^2+y^2\right]^{\frac{3}{2}}} = -k_e \lambda \left(\frac{-1}{\sqrt{z^2+y^2}}\right)\Bigg|_{x_1=0}^{x_1=l}$$

$$E_x(x,y) = k_e \lambda \left[\frac{1}{\sqrt{(x-L)^2 + y^2}} - \frac{1}{\sqrt{x^2+y^2}}\right]$$

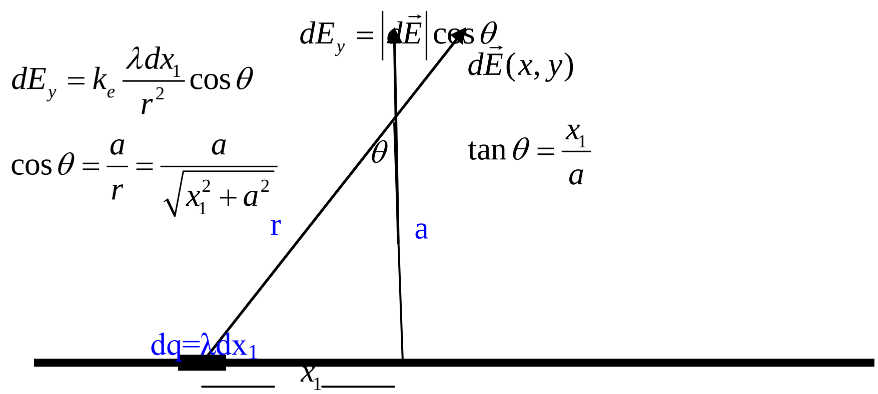
(23.22)

$$E_y(x,y)=k_e\lambda\int_0^l \frac{(y-y_1)\,dx_1}{\left((x-x_1)^2+(y-y_1)^2\right)^{\frac{3}{2}}}=(\text{with } y_1=0)=k_e\lambda\int_0^l \frac{y\,dx_1}{\left((x-x_1)^2+y^2\right)^{\frac{3}{2}}}=\text{with } z=x-x_1;\, dz=-dx_1$$

$$k_e\lambda\int_0^l \frac{y(-dz)}{\left(z^2+y^2\right)^{\frac{3}{2}}}=-k_e\lambda y\left(\frac{z}{y^2\sqrt{z^2+y^2}}\right)\Bigg|=-\frac{k_e\lambda}{y}\left(\frac{x-L}{\sqrt{(x-L)^2+y^2}}-\frac{x}{\sqrt{x^2+y^2}}\right)$$

$$Lookup:\int\frac{dz}{\left(z^2+y^2\right)^{\frac{3}{2}}}=\frac{z}{y^2\sqrt{z^2+y^2}}$$

23.5b Example 2 (infinite line): Calculate the electric field at a perpendicular distance "a" from an **infinitely long line with linear charge λ.**
Because of symmetry only the y-components of the electric field will contribute to the final result. Every positive x component will be cancelled by a negative x component. Therefore:



Note: In some textbooks the integration variables are called x, y, z. This can give rise to confusion because we calculate the electric field at the point (x, y). However, as we are dealing with definite integrals the **integration variables disappear in the final result**. You just need to keep account of your variables. For this reason alone it is of advantage to make a drawing and clearly identify your variables.

$$dE_y=k_e\frac{\lambda dx_1}{r^2}\cos\theta=k_e\frac{\lambda dx_1}{\left(x_1^2+a^2\right)}\frac{a}{r}=k_e\frac{\lambda a dx_1}{\left(x_1^2+a^2\right)^{\frac{3}{2}}}$$

(23.23)

$$E_y=k_e\int_{-\infty}^{+\infty}\frac{\lambda a dx_1}{\left(x_1^2+a^2\right)^{\frac{3}{2}}}=2k_e\int_0^{+\infty}\frac{\lambda a dx_1}{\left(x_1^2+a^2\right)^{\frac{3}{2}}};$$

look up the antiderivative: $\int \frac{dx_1}{\left(x_1^2+a^2\right)^{\frac{3}{2}}} = \frac{x_1}{a^2\sqrt{x_1^2+a^2}}$

(23.24)
$$\lim_{x_1\to 0}\frac{x_1}{a^2\sqrt{x_1^2+a^2}} = 0;\ \lim_{x_1\to\infty}\frac{x_1}{a^2\sqrt{x_1^2+a^2}} = \frac{1}{a^2} \Rightarrow \int_0^{\infty}\frac{dx_1}{\left(x_1^2+a^2\right)^{\frac{3}{2}}} = \frac{1}{a^2}$$

(23.25)
$$E_y(y=a) = +\frac{2k_e\lambda}{a} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{a}$$

The final result is only dependent on the y value (here fixed as y=a, to avoid confusion.)
Another way to integrate this:

(23.26)
$$dE_y = k_e\frac{\lambda dx_1}{x_1^2+a^2}\cos\theta$$

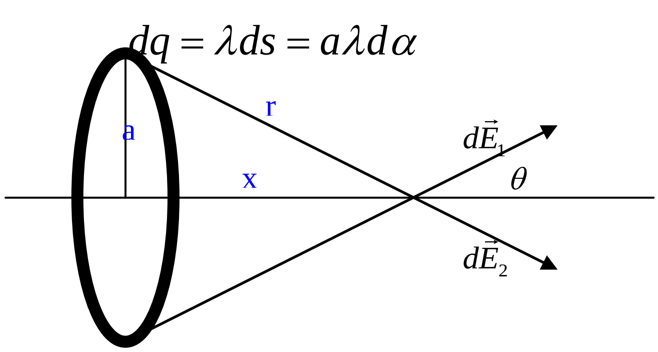
(23.27)
$$\tan\theta = \frac{x_1}{a};\ x_1 = a\tan\theta;\ dx_1 = a\sec^2\theta d\theta$$
$$x_1^2 + a^2 = a^2\tan^2\theta + a^2 = a^2\left(1+\tan^2\theta\right) = a^2\sec^2\theta$$

(23.28)
$$dE_y = k_e\frac{\lambda dx_1}{x_1^2+a^2}\cos\theta = \frac{k_e\lambda a\sec^2\theta\cos\theta d\theta}{a^2\sec^2\theta} = \frac{k_e a\lambda\cos\theta d\theta}{a^2}$$

The integral is symmetrical around the y-axis. So we just integrate from 0 to π/2 and multiply the result by 2. The angle θ varies from 0 to π/2.

(23.29)
$$E_y = 2\int_0^{\frac{\pi}{2}} dE_y = \int_0^{\frac{\pi}{2}}\frac{2k_e a\lambda\cos\theta d\theta}{a^2} = \frac{2k_e a\lambda}{a^2}\sin\theta\Big|_0^{\frac{\pi}{2}} = \frac{2k_e a\lambda}{a^2}(1-0) = \frac{2k_e\lambda}{a}$$

23.5c Example 3 (Charged ring): Calculate the electric field created by a uniformly charged ring of radius "a" at a location on the axis perpendicular to the ring.
We calculate the field on the x-axis and place the surface of the ring perpendicular and concentric to the x-axis.



The field components in the y direction cancel each other out. (The relationship between a, r, x is fixed. The contributions in the x-direction are the same for every element of charge dq_1. We therefore get:

(23.30)
$$E_x(x) = |E(x)| \cdot \cos\theta = \frac{k_e}{r^2}\cos\theta \int_0^{2\pi} \lambda a d\alpha = k_e \frac{2\pi a\lambda}{r^2}\frac{x}{r} = k_e Q \frac{x}{\left(x^2+a^2\right)^{\frac{3}{2}}}$$
$$2\pi a\lambda = Q$$

We can also use Cartesian coordinates directly from **Error! Reference source not found.**

(23.31)
$$E_x = k_e \int_0^{2\pi} \lambda a d\alpha \cdot \frac{x}{\left[\left(x-x_1\right)^2+\left(y-y_1\right)^2\right]^{\frac{3}{2}}} = \text{with } x_1 = 0; y_1 = a; y = 0$$
$$E_x = k_e \int_0^{2\pi} \lambda a d\alpha \cdot \frac{x}{\left[\left(x-0\right)^2+\left(0-a\right)^2\right]^{\frac{3}{2}}} = k_e \int_0^{2\pi} \lambda a d\alpha \cdot \frac{x}{\left[x^2+a^2\right]^{\frac{3}{2}}} = \frac{k_e \lambda 2\pi a x}{\left[x^2+a^2\right]^{\frac{3}{2}}} = \frac{k_e Q x}{\left[x^2+a^2\right]^{\frac{3}{2}}}$$

At the center of the ring x=0 and the field is 0. At a very large distance x, the ring behaves like a point charge.

(23.32)
$$\underbrace{\lim}_{x\to\infty} E_x(x) = k_e Q \frac{x}{\left(x^2+\cancel{a^2}\right)^{\frac{3}{2}}} = \frac{k_e Q x}{x^3} = \frac{k_e Q}{x^2}$$

If x is much smaller than a, we can neglect x in the denominator.

(23.33)
$$\underbrace{\lim}_{x\to 0} E_x(x) = k_e Q \frac{x}{\left(\cancel{x^2}+a^2\right)^{\frac{3}{2}}} = \frac{k_e Q}{a^3} x$$

If, in addition, we put a negative charge q_1 at x, we get a magnitude of force on this charge which is equal to:(23.34). We get the differential equation for a spring:

(23.35)
$$F = m\ddot{x} = -k_s x \text{ its solution is a sinusoidal function with angular frequency } \omega:$$
$$x = x_0 \cos\left(\omega t+\phi\right); \omega = \sqrt{\frac{k_s}{m}}$$

We compare this with the force acting on a negative charge placed near the center of a positively charged ring

$$F = -\frac{k_e q_1 Q}{a^3} x = -k_s x; \omega^2 = \frac{\frac{k_e q_1 Q}{a^3}}{m} = \frac{k_e q_1 Q}{ma^3}$$

This means that the charge negative charge oscillates back and forth around the equilibrium point with the frequency ω.

23.5d Example 4 (Charged disk):

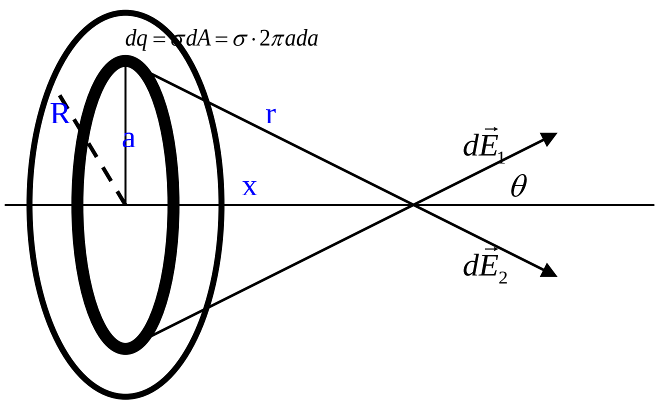
Calculate the field created by a **uniformly charged disk** of radius R with charge density σ, along the axis through its center. The exact argument of the previous example shows that only contributions to the electric field parallel to the x-axis will add up. We use the result of that example and say that the field calculated there in (23.30) is the result of a charged portion of the disk. The total charge Q becomes an infinitesimal surface charge-element. $dq = \sigma dA = \sigma \cdot 2\pi a da$

(23.36) $Q \to dq = \sigma 2\pi a \cdot da$

The integration now takes place over "a" which varies from 0 to R,

$$dE_x = \frac{k_e}{r^2} dq = \frac{k_e}{r^2} \underbrace{\cos\theta}_{\frac{x}{r}} \sigma 2\pi a da; E_x(x) = \int_0^R \frac{k_e}{\left(x^2 + a^2\right)} \underbrace{\cos\theta}_{\frac{x}{\sqrt{x^2+a^2}}} \sigma 2\pi a da =$$

$$= k_e \sigma x \pi \int_{a=0}^{a=R} \frac{2a da}{\left(x^2 + a^2\right)^{\frac{3}{2}}}$$



and we get:

setting z= $\left(x^2+a^2\right); dz = 2ada$

(23.37)
$$E_x(x) = xk_e\sigma\pi\int\frac{dz}{z^{\frac{3}{2}}} = -2xk_e\sigma\pi z^{-\frac{1}{2}}\Big| = 2xk_e\sigma\pi\left(\frac{1}{\left(x^2+a^2\right)^{\frac{1}{2}}}\right)\Bigg|_{a=R}^{a=0} =$$
$$2xk_e\sigma\pi\left(\frac{1}{x}-\frac{1}{\left(x^2+R^2\right)^{\frac{1}{2}}}\right) = 2k_e\sigma\pi\left(1-\frac{x}{\left(x^2+R^2\right)^{\frac{1}{2}}}\right)$$

23.5e Infinite Sheet of Charge.

If we let R go to infinity we are dealing with an **infinite sheet of uniform charge**. Electric field-lines diverge from both surfaces. The denominator goes to infinity, the fraction becomes 0 and the whole field turns into the simple expression 2kπσ. We use:

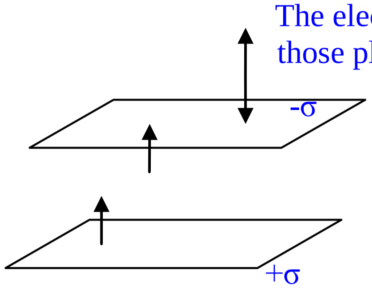
$$k_e = \frac{1}{4\pi\varepsilon_0};\ \text{with } \varepsilon_0 = 8.85\cdot 10^{-12}\,S.I.\ \text{permittivity of free space}$$

The electric field of an infinite sheet of charge is given by the constant expression

(23.38)
$$E = 2k_e\pi\sigma = \frac{\sigma}{2\varepsilon_0}$$

An often used device consists of two conducting parallel plates which have positive and negative charge surface densities respectively, which can be created through the connection to a battery.

The electric field between those plates is constant. (It is 0 outside of those plates of the parallel plate capacitor.)



Between the two sheets the positive sheet generates the field $\vec{E}_{y1} = \frac{\sigma}{2\varepsilon_0}\vec{j}$ the negative sheet generates the field E_{y2}= $\vec{E}_{y2} = \frac{\sigma}{2\varepsilon_0}\vec{j}$. The vector sum of the two fields is the upward pointing field with magnitude $\frac{\sigma}{\varepsilon_0}$.

Such two plate assemblies are convenient for experiments with charges inside a constant electric field. Inside such a capacitor we have the constant electric field:

(23.39)
$$\text{parallel plate capacitor: } E = \frac{\sigma}{\varepsilon_0}$$

23.6 Motion of a charge in a constant electric field.

If we inject electrons from the left with initial velocity **v** into such an electric field they will experience a constant downward force:

(23.40)
$$\vec{F} = q\vec{E} = -\frac{e\sigma}{\varepsilon_0}\vec{j} = m_e\vec{a}$$
$$\vec{a} = \frac{q\vec{E}}{m_e};$$

This means that we are dealing with the familiar situation of projectile motion with a downward acceleration of $\frac{e\sigma}{m\varepsilon_0}$. Everything we learnt when studying kinematic formulas and projectile motion obviously applies.

Example: Study its motion:
This is exactly like the situation where a ball is thrown horizontally. The downward force of the electric field is qE, therefore we have (directing the y-axis downward):

(23.41)
$$m\frac{d^2y}{dt^2} = qE \Rightarrow a_y = a = \ddot{y} = \frac{qE}{m} = \text{const}$$

We have the kinematic equations:

(23.42)
$$x = \mathrm{v}_{0x}t; \mathrm{v}_x = \mathrm{v}_{0x}; y = \frac{1}{2}at^2; \mathrm{v}_y = at; \mathrm{v}_y^2 = 2ay$$
$$a = \frac{qE}{m}$$

A typical value for an electric field would be 200 N/C. The mass of an electron is 9.1E-31kg, its charge is 1.6E-19 C.