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20.1 Heat and Internal Energy U.

Definition A) Internal energy U (E_{int}) of a gas is all the energy of the system that is associated with its microscopic components –atoms and molecules- when viewed from a reference frame at rest with respect to the center of mass of the system. It includes, the averages of the kinetic energies of translation and rotation, as well as the energies of vibration, potential energies inside and between molecules. The potential energies of bonding inside and outside the molecules is sometimes referred to as bonding energy. I use the symbol U for all the internal energies.

Definition B) Heat Q is defined as the transfer of energy across the boundary of a system due to a temperature difference between the system and its surroundings.

Both internal energy U and heat Q are functions of the variables P for pressure, V for volume, T for temperature, and n for number of moles. Usually, the number of moles is fixed and under such circumstances we have:

$$(20.1) \quad U = U(P, V, T) \quad \text{and} \quad Q = Q(P, V, T)$$

We talk about a PVT space in analogy to the xyz space in mechanics.

Units of Heat: calorie, Joule, Btu

$$(20.2) \quad \begin{array}{l} \text{define: } 1 \text{ cal} \hat{=} \text{ amount of energy transfer (heat) necessary to raise} \\ \text{the temperature of 1 gram of water from } 14.5^\circ\text{C to } 15.5^\circ\text{C.} \\ 1 \text{ Btu} \hat{=} \text{ amount of energy transfer (heat) necessary to raise} \\ \text{the temperature of 1 pound (lb) of water from } 63^\circ\text{F to } 64^\circ\text{F.} \end{array}$$

1Btu (British thermal unit) = 252 cal

$$(20.3) \quad \begin{array}{l} 1 \text{ cal} \equiv 4.186 \text{ J} \\ 1 \text{ Cal} \equiv 1 \text{ foodcalorie} = 1 \text{ kcal} \end{array}$$

20.2 Specific Heat and Calorimetry.

c = specific heat Q necessary to raise the temperature of 1 gram of a substance by 1°C .

$$[c] = \frac{\text{cal}}{\text{g} \cdot \text{C}^\circ} \quad \text{or} \quad \frac{\text{J}}{\text{kg} \cdot \text{C}^\circ}$$

The specific heat of water is 1, i.e. it takes 1cal to heat 1 gram of water by 1 degree Celsius. It takes 4.186 kJ to heat 1 kg of water by 1 degree Celsius.

This means that in order to increase the temperature of a mass m of a certain substance with specific heat c we need the amount of heat transfer given by

$$(20.5) \quad Q = mc\Delta T$$

To get the rate of heat change per time we divide by the time Δt , or with infinitesimal quantities:

$$\text{Power} = \frac{dQ}{dt} \equiv \dot{Q} = mc\dot{T}$$

Heat-transfer per second, rate of heat transfer.

(20.6)

It is very convenient here to use the \dot{T} notation,
rather than $\frac{dT}{dt}$!

This heat flow does not require the motion of substance and is similar to the flow of energy in a soundwave. **Heat flows always from the hotter to the colder body**; this is heat **conduction**. Don't confuse this situation with heat **convection**. From the earlier example about the anomaly of water we know that we must consider the density of liquids and gases in their movements. Hot gases rise because of their lower density. In such cases actual material exchanges take place. In heat conduction it is the vibrational disturbance that travels through an object and from object to object, in ways similar to the movement of sound.

This formula (20.5) is correct for solids and liquids. For gases we will distinguish between specific heat at constant pressure c_p and at constant volume c_v .

The amount of heat leaving object A, whose temperature goes down $\Delta T < 0$ must be equal to the amount of heat received by object B, whose temperature goes up $\Delta T > 0$.

Problem 1: A 1.50kg horseshoe originally at 600 C is dropped into an aluminum ($c_{Al}=0.215$ cal/gC $^{\circ}=900$ J/kgC $^{\circ}$) bucket containing 20.0 kg of water at 25.0 C; the mass of the aluminum bucket is 1.20 kg. The final temperature of the system is 30.0 C. Find the specific heat of the horse shoe. You can also write down that the sum of all the energy transfer of every component of the system is 0:

$$\sum Q_i = \underbrace{m_1 c_1 (T_f - T_{1i})}_{\text{horse shoe}} + \underbrace{m_2 c_2 (T_f - T_{2i})}_{\text{water}} + \underbrace{m_3 c_3 (T_f - T_{3i})}_{\text{bucket}} = 0$$

(20.7)

$$1.5\text{kg} \cdot c_h (T_f - 600^{\circ}\text{C}) + 20\text{kg} \cdot c_w (T_f - 25^{\circ}\text{C}) + 1.2\text{kg} \cdot c_{Al} (T_f - 25^{\circ}\text{C}) = 0$$

We can also solve such problems by writing down energy conservation in the form: energy lost by the horse shoe = energy gained by the water plus bucket. Make sure that both sides of the equation end up with positive numbers.

$$-1.5\text{kg} \cdot c_h (T_f - 600) = 20\text{kg} \cdot 4.186\text{kJ} (T_f - 25) + 1.2\text{kg} \cdot 900\text{J} (T_f - 25)$$

(20.8)

$$T_f = 30\text{C}$$

$$c_h = \left(\frac{84.8T - 2120}{-1.5T + 900} \right) = \frac{424}{855} = 496 \frac{\text{J}}{\text{kg} \cdot \text{C}^{\circ}} = 185 \frac{\text{cal}}{\text{kg} \cdot \text{C}^{\circ}}$$

Problem 2: A 15.0 gram lead bullet at 0 $^{\circ}\text{C}$ with a velocity of 350 m/s hits a solid brick wall. If all the energy of the bullet is changed into heat, by what amount will the temperature of lead increase? $c_{\text{lead}}=0.030$ cal/gC $^{\circ}$; $c_{\text{iron}}=0.107$ cal/gC $^{\circ}$. The melting point of lead is at 327.3 $^{\circ}\text{C}$, its latent heat of melting is 24.5 J/g = 5.85 cal/g

$$\frac{1}{2}mv^2 = Q = 919\text{J} = 220\text{cal}$$

It takes 147 cal to heat all the lead up to the melting point.

We have 72 cal left for the latent heat of melting. To melt all the lead we need 88 cal, therefore only 72/88 % of the lead will melt, i.e. 12 grams.

Thus we end up with a lump of lead at 327 °C 12 grams of which will be in molten form.

20.3 Latent Heat.

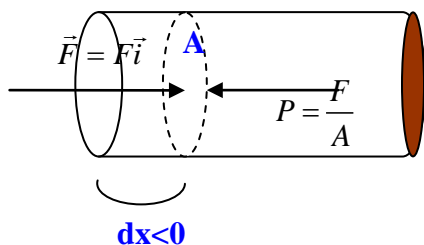
A phase change involves energy, latent heat. Energy flows for example into a solid block of ice without that the temperature of the ice water mixture changes until all the ice has melted. What is required is the latent heat of melting or fusion (for ice at 0°C: 80 cal/g). The same happens at the boiling point, where we have the latent heat of vaporization (for water at 100°C: 540 cal/g).

$$(20.9) \quad \begin{aligned} Q_f &= mL_f; L_f \text{ is the latent heat of fusion} \\ Q_v &= mL_v; L_v \text{ is the latent heat of vaporization} \end{aligned}$$

Problem 3: Calculate the heat required to have 1 mole of water evaporate at 100°C. The phase of the water is originally solid, at a temperature of -35°C. $c_{\text{ice}} = 0.50 \text{ cal/g}^\circ\text{C}$.

20.4 Work and Heat in Thermodynamics.

Imagine a cylinder of cross-section A filled with a gas at a certain pressure P with a piston tightly fitted into it. The cylinder is closed at the other end. When an outside force is applied to slowly compress the gas in the cylinder, work is done on the gas in the cylinder and therefore the internal energy of the gas must increase. The applied force on the outside of the cylinder must be barely larger than the force due to the pressure on the inside.



$$(20.10) \quad dW = \vec{F} \cdot d\vec{r} = -P \cdot A \cdot dx = -P \cdot dV$$

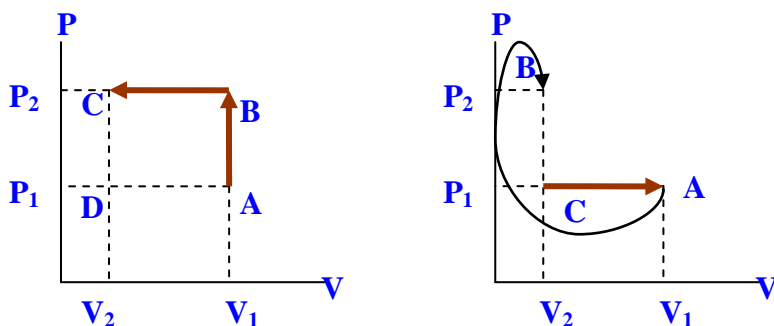
Note that we are looking at the compression from the point of view of the gas. The gas exerts a positive force F on the inside walls of the cylinder. Work is done against it when the piston compresses

the gas in the cylinder with a force \vec{F} , infinitesimally larger than the inside force of the gas on the piston wall. $\vec{F} \cdot d\vec{r} = -F \cdot dx$ results in a negative change for the gas volume. **Therefore the work done to the gas must be a positive number with the definitions below.** We assume as usual that the piston moves without being accelerated, which for a thermodynamic system is called quasi-static.

(20.11)

$$W = \int dW = \int \vec{F} d\vec{r} = - \int P A dx = - \int_{V_1}^{V_2} P \cdot dV$$

The variables for this integration are pressure P and volume V , both of which change during the compression of the gas. From the ideal gas law, we know that the temperature can also change during such a process. The coordinates for these processes are usually P and V . From the definition of work in (20.11) we conclude that the work done is the area under an oriented curve of a PV -diagram. This integral is actually a path integral, because we must specify the path on which the work is being done. We can see that the integral does depend on the path.



If we move along ABCD the work done on the gas is positive and equal to the area below the BC line. If we move from A to D directly, the work is equal to the area below the AD line, the work is again positive, but less than in the first case.

The work done in the second graph is **equal to the area contained by the curve defined by the points CAB**. If we move from C to A only along the horizontal line, the change in volume is positive, and therefore, the work done **on** the gas is negative. (The work done **by** the gas is positive.)

As we have seen above, the transfer of heat energy Q in and out of a (gas) system is another way of changing the internal energy U of a system. That process also depends on the way the heat transfer takes place. This is obvious if you consider immersing a glass container with a gas into an environment at a different (for example higher) temperature. The gas will try to expand and, if that cannot be done, its pressure will increase (or both).

One way of looking at thermodynamics is to find quantities whose change does not depend on the path taken in a PVT space, but only on the endpoints. In mechanics, we had such a quantity in the potential energy of a conservative force.

The internal energy $U(P, V, T)$ of a gas is such a quantity. The textbook by Serway uses the expression E_{int} for this energy, but we are using the term U , just like the potential energy in mechanics. But keep in mind, that the internal energy of a gas has nothing to do with its potential energy as defined in mechanics. The choice of the symbol U serves as a reminder that the internal energy U is path independent in the PVT space. We saw a moment ago that the **work done on a gas is path-dependent**. So is the heat transfer Q . Both of these quantities depend on the path taken, i.e. on the exact way in which work is being done, and heat is being transferred.

Definition of terms:

(20.12)

$$\text{Internal energy} \hat{=} \underbrace{U(P, V, T)}_{\substack{\text{Our symbol,} \\ \text{used also in most} \\ \text{other textbooks.}}} \equiv \underbrace{E_{\text{int}}(P, V, T)}_{\text{Serway}}$$

To illustrate that work and heat transfer depend on the path let us consider two processes in which we allow the volume of a gas to double, while maintaining the temperature.

- In the first case we use a piston inside a cylinder which expands the volume in what is called a **quasi static** process. Such a process can be considered a very slow process, in which infinitesimal changes of pressure, volume, and temperature occur. The bottom of the cylinder is in contact with a heat reservoir. The sides are perfectly insulated. The piston at the top is originally held down by an exterior agent. When that force is removed the piston is allowed to move upward due to the pressure of the gas inside of the cylinder. We stop the process when the volume has doubled. As there is pressure acting on the piston, the system does negative

work. $W = -\int_{V_1}^{V_2} P dV$ Energy leaves the system in form of work. Heat Q flows into the system at

the contact with the heat reservoir which is at the same initial temperature as the gas. In other words, just the right amount of heat enters the system to balance the loss of energy due to the

work done by the gas. $W = -\int_1^2 P dV = -\int_1^2 \frac{nRT}{V} dV = -nRT \ln 2$; $Q = ?$

- In the second case we can achieve the same initial and final conditions in a **free adiabatic expansion** of the gas: by removing a membrane in a cylinder (which is insulated on all sides, top, and bottom) which originally consisted of a compartment of volume V_1 separated from another evacuated compartment with the same volume. When we punch a hole into the membrane, the gas expands to double the volume. No work is being done because there is no pressure acting on a piston during the free expansion (this is called **free expansion**). No heat can flow into the system (this is called **adiabatic**). We measure no temperature change. This illustrates that work depends also on the way the expansion occurs, in other words, work is path dependent. Even though the initial and final states of both cases are the same, work and heat vary during the first process but don't in the second process. (See fig 20.6 in Serway 7.)

20.4a Review of path integrals from mechanics can be found in the file [230 ch19supp2 Vector operators.doc](#), which you can find on my website.

Here are some of the highlights:

Definition of work for a variable force on a variable path:

$$(20.13) \oint_{\text{path from A to B}} \vec{F} \cdot d\vec{r} = W = \oint_{\text{path from A to B}} F_x dx + F_y dy + F_z dz$$

This integral is a path integral and depends in general not only on the initial and final point but also on the path taken by the force F .

In the previous case of the spring force we had only one component of the force in the x-direction and the force had only the variable x. The other example was that for the force of gravity on the surface of the earth. The integral in such and similar cases never depends on the path, but only on the initial and final point. Work, in these cases, is therefore always a simple integral.

To have an illustration for a case in which the work is indeed path-dependent consider the following force:

$$\vec{F} = 3xy^2\vec{i} + 4x^2y\vec{j} \text{ from } (0,0) \text{ to } (2,1)$$

(20.14)

path 1) $y = \frac{1}{2}x$

path 2) from (0,0) to (2,0) and then from (2,0) to (2,1)

path 3) $y = \frac{1}{4}x^2$

We get the following results: path 1) 7, path 2) 8, path 3) $24/7 + 16/3 = 8.76$

For a path independent integral the vector function is the gradient of a scalar function. For a conservative force, this means that:

$$(20.15) \quad \vec{F} = -\overrightarrow{\text{grad}}U \equiv -\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle U \equiv -\frac{\partial U}{\partial x}\vec{i} - \frac{\partial U}{\partial y}\vec{j} - \frac{\partial U}{\partial z}\vec{k}$$

$$\oint_{\text{a to b along C}} \vec{F}(x, y, z) \cdot d\vec{r} = \oint_{\text{a to b along C}} \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz =$$

$$\oint_{\text{a to b along C}} \overrightarrow{\text{grad}}U \cdot d\vec{r} = \int_a^b dU = U(b) - U(a)$$

20.5 The first law of thermo-dynamics.

One way of expressing the first law of thermodynamics is to say that the change of *internal energy* ΔU of a gas system is *path independent*. It depends (just like the potential energy in gravitation, for example) only on the initial and final points of the process, as defined by the variables P (pressure), V(volume), and T(temperature.) We will see that for an ideal gas the internal energy depends only on the temperature.

ΔU is the sum of the work W done on the system as defined above and the heat Q received by the system.

(20.16)

$$\Delta U = Q + W = \int_{U_1}^{U_2} dU = \int_{U_1}^{U_2} \delta Q + \delta W = U_2 - U_1$$

$$\underbrace{dU}_{\text{total differential}} = \underbrace{\delta Q}_{\text{not a total differential}} + \underbrace{\delta W}_{\text{not a total differential}}$$

Notation: When we write ΔU this means a finite difference in energy, composed by heat Q added to the system and work W done on the system. The symbol dU indicates an infinitesimal quantity of internal energy which is a total differential, composed by infinitesimal quantities of heat δQ and work δW which are not total differentials.

The change in internal energy ΔU is a quantity which is entirely determined by the state variables P, V, T . The total derivative dU follows the same rules which we have used for the potential energy in mechanics.

The same is not true for the functions Q and W . They do not have total derivatives, which is why we must use expressions like δQ and δW .

For dU we have:

$$(20.17) \quad dU = \frac{\partial U}{\partial P} dP + \frac{\partial U}{\partial V} dV + \frac{\partial U}{\partial T} dT$$

Reminder: We have learnt that this expression can be written as the scalar product between two vectors.

$$(20.18) \quad dU = \left\langle \frac{\partial U}{\partial P}, \frac{\partial U}{\partial V}, \frac{\partial U}{\partial T} \right\rangle \langle dP, dV, dT \rangle$$

Any function that can be written as the functions in (20.18) is path-independent, when integrated over a path defined in its respective space (PVT) space or (xyz) space.

Because the internal energy U is a true total differential the energy change over a complete cycle is 0. Many processes in thermodynamics involve such cycles like in a gasoline or steam engine. For any such cyclical processes we have:

$$(20.19) \quad \Delta U = 0 \Rightarrow W = -Q \text{ for any cyclic process}$$

In a PV-diagram such a cyclic process is represented by a closed curve, and consequently, the enclosed area represents the work done on the system.

The change in internal energy is 0, which means that the same amount of energy must leave the system as Q . ($-Q$)

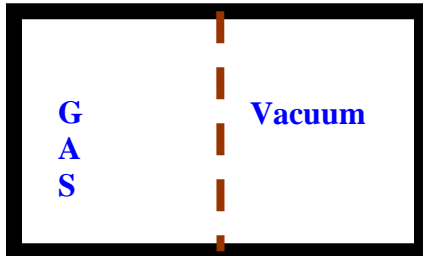
20.6 Adiabatic and Isothermal Processes.

Definitions: An adiabatic process in thermodynamics is a process in which no heat exchange occurs across the boundaries of the system.

$$(20.20) \quad Q = 0 \Rightarrow \Delta U = W \text{ for an adiabatic process}$$

Such a process is (approximately) possible by perfectly insulating a system, or by performing a process very rapidly, such that a heat exchange is negligible (car engines). (Heat exchange takes time!)

20.6a Analysis of a free and adiabatic expansion of an ideal gas. (See the earlier discussion of this process on page 6.)



Another interesting process is the free and adiabatic expansion of a gas. The term free refers to the fact that there is no work done by or on the expanding gas. (The expansion is a consequence of the random motion of the gas molecules.)

A situation like this can be experimentally accomplished by having two outwardly insulated containers connected to each other by means of an initially closed valve. One of the containers is filled with an ideal gas in thermal equilibrium. The other container is empty, it is a vacuum. If we open the valve, gas from container A streams into container B.

We wait until thermal equilibrium is established and *verify experimentally that the temperature inside the expanded gas has not changed.*

Analysis:

We know that obviously the volume of the gas has changed, but there is no piston that would do work on the gas, therefore no work is being done: $W=0$. Additionally, we know that in an adiabatic process the heat exchange is 0; $Q = 0$. We can therefore write:

$$(20.21) \quad \Delta U = \underbrace{Q}_{=0} + \underbrace{W}_{=0} = 0; \Delta V > 0$$

The ideal gas law $PV=nRT$ contains three variables. So, theoretically, the internal energy could depend on any or all of these variables. Using this law we can express pressure as a function of volume and temperature. *Experiment shows that the temperature does not change in a free expansion like the one above.* A change in volume (pressure) does occur, but does not affect the internal energy. Also, a change in pressure in the two individual compartments occurs. So, pressure and volume change, the temperature does not. The internal energy also does not change, in spite of the change in P and V. From this we can conclude or at least make the hypothesis that

$$(20.22) \quad \boxed{\begin{array}{l} \text{The internal energy of an ideal gas depends only on the temperature } T. \\ \text{If the temperature does not change the internal energy does not change.} \end{array}}$$

The change in pressure and the change in volume do not effect a change in the internal energy, thus, the internal energy (most likely) depends only on the remaining variable T. (We know that U can only be a function of P, V, and T.)

It is interesting that we can arrive at this conclusion through only observing the macroscopic variables P, V, and T, (of course using the ideal gas law) and the first law of thermodynamics. We will arrive at the same conclusion by using a microscopic statistical model for the ideal gas.

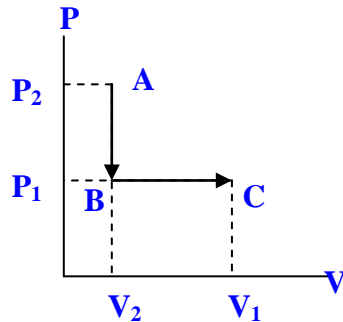
Let us study the thermodynamic functions U, W, and Q for various processes:

Isobaric process (“iso” = same, “baric” relates to “pressure”):

An isobaric process is a process that occurs at the same pressure. (B to C, for example.)

This means that in the integral for work, the function P is a constant:

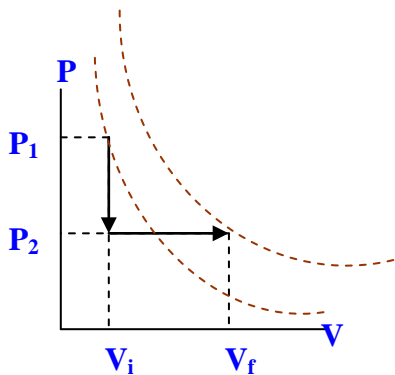
$$(20.23) \quad W = - \int_{V_i}^{V_f} P dV = -P(V_f - V_i)$$



20.6b An iso-volumetric process (isochoric) occurs without change of the volume (A to B). This means of course, that no work is being done, which, in turn, means that the internal energy change is equal to the heat exchange.

(20.24)

$$\Delta U = Q$$



20.6c Iso-thermal process: The second most important process after an adiabatic process is an iso-thermal process.

As the name indicates, an iso-thermal process occurs at the same temperature T. Note that in a PV diagram, there is no variable for T. We will later find how for some simple processes T is related to pressure and volume. We always have the ideal gas law, to connect the three variables. Generally, one can say that the *iso-thermal lines* in a PV-diagram can be represented by hyperbolic curves like so:

The curves obviously never intersect. We later introduce also adiabatic lines on the same PV-diagram and we shall see that they are represented by steeper lines. We can calculate the work done in isothermal processes.

20.6d Work during an isothermal Expansion of an Ideal Gas.

$$PV = nRT \Rightarrow T = \text{constant} \Leftrightarrow PV = \text{constant}$$

(20.25)

$$P = \frac{nRT}{V}$$

Therefore, we can express P through V in our work integral:

$$(20.26) \quad W = - \int_{V_i}^{V_f} P dV = - \int_{V_i}^{V_f} \frac{nRT}{V} dV = nRT \ln \frac{V_i}{V_f}$$

To summarize, in adiabatic processes $W = \Delta U$ and in isothermal processes $W = nRT \ln \frac{V_i}{V_f}$

20.7 Energy Transfer Mechanisms :

a) **Thermal Conduction**, $\overrightarrow{grad}T \equiv \overrightarrow{\nabla} \cdot T$.

Wherever there is a temperature difference we can define the gradient of temperature, which is the temperature change per distance in every direction. Whenever and wherever such differences exist there will be a flow of heat in the direction of lower temperature, like in the direction of $-\frac{dT}{dx}$. This flow of

heat is described by a current density $\vec{j}(x,y,z)$ flowing through an area

For an infinitesimal thin, one-dimensional slab of isotropic material the rate of heatflow is given by

$$(20.27) \quad Power = \frac{dQ}{dt} \equiv \dot{Q} = kA \left| \frac{dT}{dx} \right| = -kA \frac{dT}{dx} \text{ or } \dot{Q} = -k\overrightarrow{A} \cdot \overrightarrow{grad}T(x, y, z)$$

$$k \hat{=} \text{thermal conductivity; material constant in } \frac{\text{Watts}}{\text{mC}^\circ}$$

To get a positive number for power, even though ΔT is negative (heat flows from hot to cold), we have written

$$(20.28) \quad \frac{dQ}{dt} = Power = -kA \frac{dT}{dx} \text{ or } -k\overrightarrow{A} \cdot \overrightarrow{grad}T(x, y, z)$$

(as long as T is constant over the surface A)

Temperature changes from point to point, and during time. T is therefore a function of x,y,z, and t.

In the case of a rectangular setup, you can use the Cartesian expression of the gradient

$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle T$. In the case of a cylindrical situation in which you have a radial heatflow use

cylindrical coordinates for T and the gradient :

$$(20.29) \quad \overrightarrow{\nabla} T = \frac{1}{r} \frac{\partial T}{\partial \theta} \vec{u}_\theta + \frac{\partial T}{\partial r} \vec{u}_r + \frac{\partial T}{\partial z} \vec{u}_z \text{ (See homework problem 18.)}$$

The change of energy per unit time in (20.28) is equal to the scalar product between the current density and the cross-sectional area through which the current flows. For a constant current density and a constant surface, this is simply the product:

$$(20.30) \quad \dot{Q} \equiv \frac{dQ}{dt} = \vec{j}_h \cdot \vec{A}$$

This turns into a surface integral if we allow for arbitrary surfaces:

$$(20.31) \quad \dot{Q} \equiv \frac{dQ}{dt} = -k \iint_A \overrightarrow{grad}T(x, y, z) \cdot d\vec{A}$$

In differential form this is equal to:

$$(20.32) \quad \vec{j}_h = -k \cdot \overrightarrow{\text{grad}T}$$

20.7 add. Current densities:

(Now, current densities j have the dimensions of a current I divided by the cross-section "A" through which the current flows. The **heat current density** j_h flowing through a surface results in the **heat current** $I = dQ/dt$, which has the units of energy per time, i.e. power. A liquid current density j is defined as the liquid density times its velocity field vector: $\vec{j} = \rho \vec{v}$. Multiplying current density j with the cross-section A (as a scalar product) results in $\vec{j} \cdot \vec{A} = \rho \vec{v} \cdot \vec{A}$. The dimensional analysis shows that this expression has the dimensions of mass/time, just like the heat current has the dimension energy per time which is equal to power. See Addendum of the last pages of this paper.)

For a rectangular bar of cross-section A and length L, whose ends are maintained at constant (different) temperatures, the heat flow per second is given by :

$$(20.33) \quad \text{Power} = kA \frac{(T_{hot} - T_{cold})}{L} = -kA \frac{(T_{cold} - T_{hot})}{L}$$

$$\frac{\Delta T}{\Delta x} < 0$$

It takes a longer time for the heat to flow through a longer bar, therefore the power is inversely proportional to the length. More heat flows at the same time during a larger cross section A, therefore power is directly proportional to A.

For a compound slab containing several materials of thicknesses L_i and thermal conductivities k_i , the rate of energy transfer through the slab at steady state conditions is given by:

$$(20.34) \quad \text{Power} = A \frac{(T_{hot} - T_{cold})}{\sum_i \frac{L_i}{k_i}}$$

For two slabs connected through the same cross-section but different lengths and thermal conductivities we can easily verify the equation (20.34).

The power flowing through the first slab equals the power flowing through the second slab. They

share the same temperature at the point of contact. $P_1 = \frac{k_1}{L_1} A (T_h - T) = \frac{k_2}{L_2} A (T - T_c) = P_2$ Solving

for T and inserting in P_1 yields:

$$\text{Power} = A \frac{(T_{hot} - T_{cold})}{\frac{L_1}{k_1} + \frac{L_2}{k_2}}$$

Problem 4:

Compare the heat flow through a window pane of thickness 8.00 mm to the flow across a double pane window with two glass panes of thickness 4.00 mm but separated by a 4.00 mm space filled with a gas, say nitrogen. The heat conductivity constants are as follows, in W/m°C: glass $k_{glass} = 0.800$

W/m°C; nitrogen or air $k_{air} = 0.0234$ W/m°C.

Assume a window area of 1m^2 and a temperature difference of 20.0C° .

The heat flow through the simple pane is obviously $\frac{0.8}{0.008} 20 = 2000W = 2kW$

The ratio between the powers is $\frac{k_g / 2L}{\frac{1}{\frac{L}{k_{air}} + 2 \frac{L}{k_{glass}}}} = \frac{0.8 / 0.008}{\left(\frac{0.004}{0.0234} + \frac{0.008}{0.8} \right)^{-1}} = \frac{100}{(0.18)^{-1}} = 18$ (36 for a

triple pane window)

In other words, the double pane window is 18 times more efficient as an insulator than the simple pane window with the same thickness of glass. Instead of losing 2kW, it only loses 111 Watts.

A triple pane window with three 4 mm panes and two 4mm airspaces loses only 55 Watts. It is only twice as efficient as a double pane window. The major gains in energy conservation come from the transition from single pane windows to double pane windows. If you use aluminum framed windows (thermal conductivity 238!!!!) compared to wood framed windows ($k_{wood}=0.08$) you might as well have holes in the wall. Aluminum is 3000 times less efficient as an insulator than wood. Also note that wood is still ten times more efficient than glass. Note also that concrete (stone) has the same coefficient as glass.

Problem 5: Calculate the temperature at the junction of two bars, one of steel, the other of copper. $k_{steel}=79.5$ W/mK; $k_{Cu}=397$ W/mK. Both bars have the same length of 50.0 cm, and cross-section of 12.0cm^2 .

b) Heat Transfer through Convection :

Heat energy transferred by a movement of a warm material substance (gases, liquids) is called convection. The movement of actual particles with kinetic energy is involved in this process. Those particles move in the direction of lower particle concentration, called diffusion, counter acted by gravitaion. Convection depends on the shape of the heat emitter, for example pipes, spheres, boxes.

c) Heat Transfer through Radiation. (electromagnetic waves):

(20.35)

Stefan's law : $Power = e\sigma AT^4$

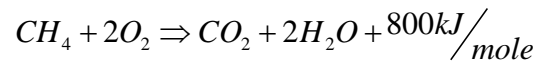
$\sigma = \text{Stefan-Boltzmann constant} = 5.669 \times 10^{-8} \frac{W}{m^2 K^4}$

Problem 6: The sun has a surface temperature of 5700 °K. If its emissivity constant e is equal to 1, calculate its power output. The radius of the sun (a hydrogen gas ball) is $7E8$ m. The earth's orbital radius around the sun is $1.5E11$ meters. Calculate the intensity of the electromagnetic (em) radiation that reaches the surface of the earth's atmosphere.

20.7d Estimates of energy output in the daily use of crude oil:

Problem 6:

Energy (heat) created by the burning of 100 million bbl of oil per day.



(A partial measure of the internal energy of a system is enthalpy H. **Enthalpy** cannot be directly measured, but changes in it can be, just like in the case of internal energy U. If the outside pressure on a system is held constant, a change in enthalpy entails a change in the system's internal energy, plus a change in the system's volume (meaning the system exchanges energy with the outside world). For example, in endothermic chemical reactions, the change in enthalpy is the amount of energy absorbed by the reaction; in exothermic reactions, it is the amount given off. $H=U+PV$. See additional definitions in chapter 22.)

Approximations used: 1 barrel of oil =42 US gallons=159 liters; 1US gallon=3.876Liters.

Average Oil density=800kg/m³

Average mass of oil per barrel

$$M = 0.159m^3 \cdot 800 \frac{kg}{m^3} \approx 127kg$$

Average energy content of oil =12kcal/g =12Mcal/kg=48MJ/kg

$$E / bbl = 48 \frac{MJ}{kg} \cdot \frac{127kg}{bbl} = 6.1 \cdot 10^3 \frac{MJ}{bbl} = 6.1 \cdot 10^9 \frac{J}{bbl}$$

Problem 7: Energy per bbl:

Average energy transformed per day due to the usage of 100 million barrels of oil per day is equal to

$$6.1 \cdot 10^{17} J / day \text{ which corresponds to a power output of } \frac{6.1 \cdot 10^{17} J}{24 \cdot 3600 s} = 7.1 \cdot 10^{12} \frac{J}{s}$$

Let us compare this power to the average power delivered by the sun to the earth. The intensity of the sun's radiation at the outer atmosphere of the earth is approximately $1.4kJ/m^2$. (This energy depends on the surface temperature of the sun and is proportional to that temperature at the power of 4. The surface temperature of the sun is around 5800Kelvin.) The total energy radiated into the earth's

$$\text{atmosphere per second is equal to } 4\pi r^2 \cdot 1.4 \cdot 10^3 \frac{J}{s \cdot m^2} = 4\pi \cdot (6.37 \cdot 10^6 m)^2 \cdot 1.4 \cdot 10^3 \frac{J}{s \cdot m^2} = 7.2 \cdot 10^{17} \frac{J}{s}$$

The ratio between these two numbers is interesting. **To one significant figure it is equal to**

$$10^{-5} = \frac{7.1}{7.2 \cdot 10^5} = 10^{-5} = 10^{-3}\%$$

This number corresponds to the average relative change of the power intake of the earth's atmosphere due to the burning of oil, as compared to the heat output of 100 million barrels of oil.

For every calorie created by the burning of oil, the sun adds 100 thousand calories to the atmospheric heat content. This assumes the worst case scenario in which oil is just transformed into pure heat energy. Much of the energy is of course used to do work. Or, put inversely, the heat injected into the atmosphere due to the burning of oil is equal to 1/100,000 of the heat injected into the atmosphere by the sun.

Problem 8: The relative uncertainty in the intensity of the sun's radiation (Compare the fluctuation of the sun's power output with the heat generated by 100 million barrels of oil, per day.

based on: Power of the sun $= e\sigma AT^4 = \text{emissivity} \cdot \text{Stefan's constant} \cdot \text{surface area of the sun} \cdot (\text{absolute temperature})^4$

The relative uncertainty is $\frac{4\Delta T}{T} + \frac{\Delta e}{e} + \frac{2\Delta R}{R}$

The emissivity fluctuates, and so does the radius of the sun (a ball of super hot hydrogen gas) and the temperature. Let us assume though that the only significant variation in the power output comes from temperature changes.

If we equate the relative uncertainty in the energy output of the sun to the ratio between oil energy/sun energy we can calculate the temperature change of the sun needed to compensate for the energy fluctuation due to the burning of oil:

$$4\frac{\Delta T}{T} = 10^{-5}$$

If we use 6000K for the rounded average temperature of the Sun, we get

$$\Delta T = T \cdot 10^{-5} = 6000K \cdot 10^{-5} = 0.06K$$

In other words, a 6/100 degree change in the temperature of the sun's surface would account for all the burning of the oil on earth. (about 0.1 degree F.)

The contribution of heat to the earth's atmosphere by human activities is completely covered by the fluctuations of the sun's energy output.

20.8 Addendum (optional), The Continuity Equation:

The concepts of current density and current occur frequently in physics but in different contexts. The easiest approach is to conceive of a flowing liquid which we call current in normal language:

The amount of liquid contained in a cylindrical element is

$$(20.36) \quad dM = \rho dV = \rho A dx$$

The change in time of the mass is consequently

$$(20.37) \quad \frac{dM}{dt} = \rho A \underbrace{\frac{dx}{dt}}_{\frac{dV}{dt}} = \rho A v$$

The amount of liquid that flows through a fixed surface A, of a tube for example, can be described through (20.38)

$$\frac{dM}{dt} = \rho A \frac{dx}{dt} \text{ or } \rho \vec{A} \vec{v},$$

In our cases here the density is constant, but it could also be a scalar field, i.e. a scalar function in several variables. If you think of the density of air, it certainly changes slightly from location to location. If you open the valve of a container with compressed air, the density changes can obviously not be ignored. We would have to use $\rho(x, y, z)$ as a scalar field rather than ρ as a constant. The velocity is also a function of x, y, z, t . **This means that velocity is a so-called**

$$(20.39) \quad \vec{v}(x, y, z, t) = \langle v_x, v_y, v_z \rangle \text{ with each component of the velocity, being a function of } x, y, z, \text{ and } t.$$

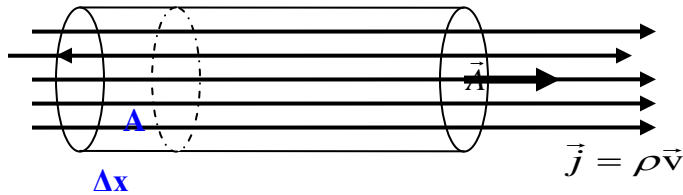
vector field. The local change of such a function can occur with respect to x, y, z , or t .

The product of density ρ times velocity \vec{v} is called current density \vec{j} . In general, this is the density of a quantity like mass, liquid, heat, electric charges, and so on:

$$(20.40) \quad \vec{j} = \rho \vec{v} = \begin{array}{l} \frac{M}{V} \vec{v} \text{ mass current density} \\ \frac{N}{V} \vec{v} \text{ number current density} \\ \frac{Q}{V} \vec{v} \text{ heat energy (Q) current density} \\ \frac{Ne}{V} \vec{v} \text{ electric charge current density} \end{array}$$

The current itself, called I , (not the current density vector \vec{j}), is a scalar, which results from the scalar product between the current density \vec{j} and the surface vector \vec{A} . The surface vector has the magnitude of the surface area \vec{A} through which the density flows and the direction perpendicular to the surface. The infinitesimal surface element $d\vec{A}$ is a vector oriented to the outside of the volume surface. This means that a mass streaming out of the volume is positive, the mass streaming into it is negative. In other words, if there is no additional mass created

inside of the cylinder, the amount streaming out of any surface equals the amount streaming in, provided that the mass density ρ does not change.



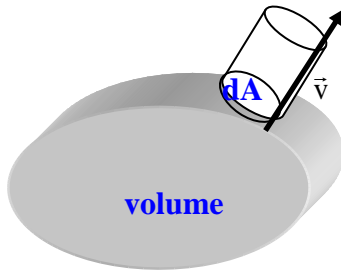
(20.41)

$$I = \vec{j} \cdot \vec{A} = \iint_{\text{surface}} \vec{j} \cdot d\vec{A}$$

integral

We see that the current is the rate of change of the quantity under consideration, for example, in the case of a liquid running out of the end of a cylindrical tube with cross-section A

$$(20.42) \quad I = \frac{M}{V} \underbrace{\vec{v} \cdot \vec{A}}_{\frac{dx}{dt} \cdot \frac{dV}{dt}} = \frac{d}{dt}(\rho V) = \frac{dM}{dt}$$



Let us prove the continuity equation for any kind of liquid or energy flow.

Calculate the outflow of liquid out of the volume V.

Let us call dA the area element through which the liquid flows. The particles which lie on the surface dA at the time t , will move by the distance $ds = v dt$ during the time dt . The amount of liquid that passes through the surface during the time dt is therefore equal to the amount of liquid contained in the cylinder created by the side $v \cdot dt$ and the base dA .

(20.43)

$$dM = \rho \vec{v} dt \cdot d\vec{A}$$

If the angle between the normal of dA and the velocity is obtuse dM turns negative. For a closed surface dM becomes negative when liquid enters the surface through dA . For the total amount of liquid entering the surface in unit time we get

$$(20.44) \quad \frac{dM}{dt} = \oiint_{A=\partial V} \rho \vec{v} \cdot d\vec{A}$$

(Note that we integrate over space, not over time. We are calculating the total amount of liquid entering the surface in unit time.) This integral is also called the net flux of the vector \vec{v} through the total and closed surface A containing the volume V within itself. We shall visit the concept of flux again in chapter 24, in the context of electric fields.

Any portion of incoming liquid would have a negative sign in this integral. The whole amount of liquid which resides in the domain V, having A as its closed surface, is given by the volume integral:

$$(20.45) \quad M = \iiint_V \rho dV$$

During the time dt this volume changes by dM/dt

$$(20.46) \quad \frac{dM}{dt} = \frac{d}{dt} \iiint_V \rho dV$$

We apply Leibniz' rule for the derivatives of an integral to the derivative of this volume integral: (The normal derivative of the integral is equal to the integral of the partial derivative of the integrand.)

$$(20.47) \quad \frac{d}{dt} \iiint_V \rho dV = \iiint_V \frac{\partial \rho}{\partial t} dV$$

We are in the process of calculating the outflow of liquid out of the volume V, which means that the amount in (20.47) is negative. This amount is then equal to (20.44):

$$(20.48) \quad \oiint_{\partial V} \rho \vec{v} \cdot d\vec{A} = - \iiint_V \frac{\partial \rho}{\partial t} dV$$

Gauss' theorem for surface integrals states that the surface integral of a vector function \vec{E} taken over a closed surface is equal to the volume integral of the divergence of that vector function taken over the volume V, for which the surface is an enclosing boundary.

$$(20.49) \quad \underbrace{\iiint_V \operatorname{div} \vec{E} \cdot dV = \oiint_{\partial V=A} \vec{E} \cdot d\vec{A}}_{\text{Gauss' theorem}}$$

Using this theorem we express the surface integral of (20.48) by a volume integral:

$$(20.50) \quad \oiint_{\partial V} \rho \vec{v} \cdot d\vec{A} = \iiint_V \operatorname{div}(\rho \vec{v}) \cdot dV$$

We have now an equality between two volume integrals:

$$(20.51) \quad \iiint_V \operatorname{div}(\rho \vec{v}) \cdot dV = - \iiint_V \frac{\partial \rho}{\partial t} dV \text{ or}$$

$$(20.52) \quad \iiint_V [\operatorname{div}(\rho \vec{v}) + \frac{\partial \rho}{\partial t}] dV = 0$$

Remembering our math again, we apply the theorem that **if a triple integral of a continuous integrand is 0 over an arbitrary domain, then the integrand must be identically equal to 0.**

Thus, we get the extremely important **continuity equation** for the flow of an arbitrary (compressible or incompressible) liquid of density ρ and flow velocity \vec{v} :

$$(20.53) \quad \boxed{\operatorname{div}(\rho \vec{v}) = - \frac{\partial \rho}{\partial t}}$$

The density does not have to be a constant and can be a function space and time variables. The same is true for the velocity field. (Note: **We talk about a velocity field if the velocity vector can change not only in time but also from point to point in space.** This means that ρ depends on time directly as well as indirectly through the coordinates x, y, z)

As an additional annoying exercise let us assume that the density ρ is a function of x, y, z and t .

$$(20.54) \quad \frac{d\rho(x, y, z, t)}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} \frac{dx}{dt} + \frac{\partial \rho}{\partial y} \frac{dy}{dt} + \frac{\partial \rho}{\partial z} \frac{dz}{dt}$$

The partial derivatives with respect to $x, y,$ and $z,$ constitute the gradient of the density scalar function:

$$(20.55) \quad \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\overline{\operatorname{grad} \rho}) \cdot \vec{v}$$

We want to use the continuity equation and therefore expand the div expression according to the rules of vector-derivative-operators: (Both density and velocity have derivatives.)

$$(20.56) \quad \operatorname{div}(\rho \vec{v}) = \vec{\nabla} \cdot (\rho \vec{v}) = (\vec{\nabla} \rho) \cdot \vec{v} + \rho \cdot \vec{\nabla} \cdot \vec{v} = \vec{v} \cdot \overline{\operatorname{grad} \rho} + \rho \cdot \operatorname{div} \cdot \vec{v}$$

$$(20.57) \quad \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \underbrace{(\overline{\operatorname{grad} \rho}) \cdot \vec{v}}_{\operatorname{div}(\rho \vec{v}) - \rho \operatorname{div} \cdot \vec{v}} = \frac{\partial \rho}{\partial t} + \underbrace{\operatorname{div}(\rho \vec{v})}_{-\frac{\partial \rho}{\partial t}} - \rho \operatorname{div} \cdot \vec{v}$$

$$(20.58) \quad \frac{d\rho}{dt} = -\rho \operatorname{div} \cdot \vec{v} \text{ or}$$

$$(20.59) \quad \boxed{\operatorname{div} \vec{v} = - \frac{1}{\rho} \frac{d\rho}{dt}}$$

The divergence of any velocity field is equal to the negative rate of relative change of its density at any given point at a given time. If the liquid is not compressible, we get

(20.60)

$$\boxed{\text{div}\vec{v} = 0}$$

If we apply a volume integral to this expression and transform it into a surface integral we get the simple continuity equation for an incompressible fluid:

$$(20.61) \quad \iiint_V \text{div}(\rho\vec{v})dV = 0 = \oiint_{A=\partial V} \rho\vec{v} \cdot d\vec{A} = \text{Inflow-outflow}=\text{total flux}$$

The net flux of an incompressible liquid is always 0.

20.8a Continuity equation and heat flow:

We apply this information now to the flow and transport of heat energy Q.

The flow has a direction, and like in the flow of liquid, we define a density ρ_h of heat through

$$(20.62) \quad \rho_h = \frac{Q}{V}$$

and a current density \vec{j}_h for heat flow:

$$(20.63) \quad \vec{j}_h = \rho_h \vec{v}$$

Equation (20.53) becomes:

$$(20.64) \quad \text{div}\vec{j}_h = -\frac{\partial\rho_h}{\partial t}$$

For a heat current density

$$Q = mc\Delta T; \rho_h = \rho_m \cdot c \cdot \Delta T$$

(20.65) c is the heat capacity, and ρ_m is the mass density $\frac{m}{V}$

$$\vec{j}_h = \rho_h \vec{v}$$

$$(20.66) \quad \text{div}\vec{j}_h = -\frac{\partial\rho_h}{\partial t} = -\frac{\partial(\rho_m c \Delta T)}{\partial t} = -\rho_m c \dot{T}$$

20.8b Equation for heat conduction:

The flow of heat follows the gradient:

From (20.32) we know:

$$(20.67) \quad \vec{j}_h = -k \cdot \overrightarrow{\text{grad}T}$$

Combining the two formulas gives us:

$$(20.68) \quad \text{div}\vec{j}_h = \text{div}\left(-k \cdot \overrightarrow{\text{grad}T}\right) = -\frac{\partial\rho_h}{\partial t}$$

Watch out for the different densities here: we have a heat density $\rho_h=Q/V$ and a mass density $\rho_m=m/V$.

$$(20.69) \quad \rho_h = \rho_m c \Delta T$$

which is obtained by dividing the amount of heat $Q=mc\Delta T$ by the volume occupied by the substance of mass m . **Therefore,**

$$(20.70) \quad \frac{\partial \rho_h}{\partial t} = \rho_m c \frac{\partial T}{\partial t} = \rho_m c \dot{T}$$

$$(20.71) \quad \operatorname{div}(k \overrightarrow{\operatorname{grad} T}) = k \Delta T = \rho_m c \dot{T}$$
$$\overrightarrow{\nabla}^2 T = \frac{\rho_m c}{k} \dot{T}$$

The equation obtained is a partial differential equation for the temperature distribution and is called the equation for heat conduction:

$$(20.72) \quad \boxed{\Delta T = \frac{\rho c}{k} \dot{T}}$$