Work, potential energy, and path integrals, differential vector operators

The definition of work by a constant force **F** acting over a distance **r** is:

(1.1) $W = \vec{F} \cdot \vec{r} = F_x x + F_y y + F_z z = Fr\cos\theta$

Assume that we drag a crate of mass m along the floor, exerting a force **F** at an angle with the horizontal, without accelerating the crate. The horizontal force component is then equal and opposite to the force of friction. If we drag the crate from x=a to x=b, the work done by the force **F** is equal to F_x d, where d=b-a. The work done by friction is $-F_x$d.

If we lift a box of mass m up by a distance h without acceleration, the work done by the lifting force is mgh, whereas the work done by gravity is –mgh. If we drop this box, it can again do work due to gravity. This possibility is captured in the word **potential energy**: U(y). We say that the box has the potential energy U=mgy, and the potential energy change of the box while being lifted from a to b, is

(1.2) $$\Delta U = mgb - mga = mgh$$

Thus, we associate a function U(y) with this force and call it the **potential energy function U(y)** for the force m**g. A force which has a potential energy function, from which it derives, is called conservative.**

One can see easily that:

$$\vec{F} = -mg\vec{j}$$

(1.3) $F_y = -\dfrac{dU}{dy} = -mg$ The force is the negative derivative with respect to y of the potential energy function U(y).

Memorize this new law: Whenever the work done by a force leads to a fully recoverable energy, the work done by this force is equal to the negative change in the potential energy. (When a force does work, it does so at the expense of its potential energy.) We will study this in great detail in this chapter.

(1.4) $$W = -\Delta U$$

The force associated with such a potential energy is called conservative for reasons we explain later.

If we have a non-constant force **F**(x)=-k**x** in one dimension, like the spring force, and want to calculate the work done by this force while it moves an object of mass m from point A to B on the x-axis, we proceed as follows:

(1.5)
$$\text{infinitesimal amount of work: } dW_i \approx F\left(x_i\right)\Delta x_i$$
$$W \approx \sum_{i=1}^{n} dW_i \approx \sum_{i=1}^{n} F\left(x_i\right)\Delta x_i$$

If we make the Δx intervals smaller and smaller, while allowing n to go to infinity, we form an infinite sum of tiny rectangles which is equal to the integral of the function F(x) taken between the points A and B.

(1.6) $$W = \int_A^B F(x)dx = -\frac{kx^2}{2}\bigg|_A^B = \frac{-k}{2}\left(x_B^2 - x_A^2\right)$$

Following the example of a constant force with a path in one dimension we define the **potential energy** in general as the function U(x,y,z) such that **every component of the force** is the **negative partial derivative of the potential energy**. The reason for the negative can be found in the fact that the potential energy is diminished, when the force is being applied to do work. (From a purely mathematical point of view the negative sign is not necessary.)

If the force in question has more than one variable we must use the scalar product between the force and the infinitesimal displacement vector to give us the infinitesimal amount of work dW:

(1.7) $$d\vec{r} = \langle dx, dy, dz \rangle$$
$$dW = \vec{F}(x,y,z)\cdot d\vec{r}$$

Definition of work for a variable force on a variable path:

(1.8) $$\oint_{\substack{path\\ from\ \text{A to B}}} \vec{F}\bullet d\vec{r} = W = \oint_{\substack{path\\ from\ \text{A to B}}} F_x dx + F_y dy + F_z dz$$

This integral is a path integral and depends **in general** not only on the initial and final point but also on the path taken by the force **F**.
In the previous case of the spring force we had only **one component of the force in the x-direction and the force had only the variable x**. The other example was that for the force of gravity on the surface of the earth. The integral in such and similar cases never depends on the path, but only on the initial and final point. Work, in these cases, is therefore always a simple integral.

To have an illustration for a case in which the work is indeed path-dependent consider the following force:

(1.9)
$$\vec{F} = 3xy^2\vec{i} + 4x^2y\vec{j} \text{ from (0,0) to (2,1)}$$
$$\text{path 1) y=}\frac{1}{2}x$$
$$\text{path 2) from (0,0) to (2,0) and then from (2,0) to (2,1)}$$
$$\text{path 3) y=}\frac{1}{4}x^2$$

We get the following results: path 1) 7, path 2) 8, path 3) 24/7+16/3=8.76

For another example see further below (1.40).

The generalized relationship between a conservative force **F** and its potential energy U is given in terms of the differential operator (boldface **F** indicates that the function F is a vector)

The general formula for work as defined by formula (1.8) has an integrand which looks exactly like the **perfect derivative of a function**; let us call that scalar function S(x, y, z):

This function was introduced during our discussion of uncertainties. We defined the total or perfect derivative as:

$$(1.10)\quad dS = \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz$$

For example, if S (x,y,z) = $3x^2 \cdot y^3 \cdot z^{-1/2}$ we get

$$(1.11)\quad \begin{aligned} S(x,y,z) &= \frac{3x^2y^3}{\sqrt{z}} \\ dS &= \frac{6xy^3}{\sqrt{z}}dx + \frac{9x^2y^2}{\sqrt{z}}dy - \frac{3x^2y^3}{2z^{\frac{3}{2}}}dz \end{aligned}$$

Note that this S is a scalar function, not a vector.
Whenever we are faced with a path-integral containing an integrand like the one above we would immediately know how to integrate it, namely:

$$(1.12)\quad \oint\limits_{\substack{\text{a to b along} \\ \text{path C}}} \frac{6xy^3}{\sqrt{z}}dx + \frac{9x^2y^2}{\sqrt{z}}dy - \frac{3x^2y^3}{2z^{\frac{3}{2}}}dz = \oint\limits_{\text{a to b}} dS = S(b) - S(a)$$

We need to find a way to determine whether the factors in front of the dx, dy, and dz are the partial derivatives of a function S(x, y, z). This means we need to investigate whether we can write the path-integral (1.8) in the following form:

$$(1.13)\quad \begin{aligned} &\oint\limits_{\text{a to b along C}} \vec{F}(x,y,z)\cdot d\vec{r} = \oint\limits_{\text{a to b along C}} \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz = \\ &\oint\limits_{\text{a to b along C}} \overrightarrow{gradS}\cdot d\vec{r} = \int_a^b dS = S(b) - S(a) \end{aligned}$$

In order to find an easy intuitive answer to this question we introduce the concept of vector operators. They are an extension of the concept of the derivative operator $\frac{d}{dx}$. If we apply this operator to a scalar function S(x) we are to take the derivative of this function. The basic vector operator is called the del operator and is defined as: $\vec{\nabla}$ as the differential **vector** operator (del-operator, an upside down Greek Δ)

(1.14)

$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \equiv \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \equiv \left\langle \partial_x, \partial_y, \partial_z \right\rangle$ This vector operator must operate on some mathematical quantity to its right; it can be a scalar function, a vector function, or another differential operator. In the case of a vector function it can operate through a dot product $\vec{\nabla}\cdot$ or a cross-product $\vec{\nabla}\times$.

In the case that the del operator acts on a scalar function, it is called gradient or simply grad.

$$\boxed{\overrightarrow{grad}U(x,y,z) = \left\langle \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right) \right\rangle \equiv \left(\partial_x U, \partial_y U, \partial_z U \right) = \vec{\nabla} U} \tag{1.15}$$

To simplify the writing of partial derivatives one often defines that

$$\boxed{\partial_x \equiv \frac{\partial}{\partial x}; \partial_y \equiv \frac{\partial}{\partial y}; \partial_z \equiv \frac{\partial}{\partial z}} \tag{1.16}$$

Do not mix this notation up with subscripts for vector components, like for example A_x which is the x-component of the vector or vector function **A (also called vector field)**. A_x can be a function of x, y, and z. For example:

$$\vec{A}(x,y,z) = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle = \tag{1.17}$$

the vector component $A_x = \frac{x}{\sqrt{x^2+y^2+z^2}}$ is a function of x,y,z

∂_x is the partial derivative with respect to x of the function following the operator; for example, if $U(x,y) = x^2 \cdot y^3$ the operator ∂_x yields $\partial_x U(x,y) = 2x\cdot y^3$.

Reminder: The definition of a partial derivative, with respect to x for example, is obtained by treating all variables in a function except x as constants. (See the "uncertainty" paper.)

Here is a summary of the three major uses of the del operator:

$$a)\vec{\nabla} \equiv \langle \partial_x, \partial_y, \partial_z \rangle = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$b)\vec{\nabla} U = \overrightarrow{\text{grad}}\text{U gradient of U} = \left\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right\rangle; U \text{ is a scalar function U(x,y,z)}$$

$$c)\vec{\nabla} \cdot \vec{E} = div\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \text{ divergence of the vectorfield } \vec{E} = \langle E_x, E_y, E_z \rangle$$

$$\text{d)}\vec{\nabla} \times \vec{B} = curl\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ B_x & B_y & B_z \end{vmatrix} = \vec{i}\,(\partial_y B_z - \partial_z B_y) + \vec{j}\left(\partial_z B_x - \partial_x B_z\right) + \vec{k}\left(\partial_x B_y - \partial_y B_x\right)$$

(1.18) $d1$)If the vector $\vec{B}$ is a vector in two dimensions x and y the curl becomes a vector in the z-direction:

$$\vec{\nabla} \times \vec{B} = curl\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & 0 \\ B_x & B_y & 0 \end{vmatrix} = \vec{k}\left(\partial_x B_y - \partial_y B_x\right)$$

Intermezzo on determinants:
Let me emphasize some important facts about the cross product which we use here for the definition of the curl:

In the definition of the curl through the determinant we must bear in mind that the multiplication between a derivative operator and a function is not commutative, for example:

(1.19) $$\partial_y B_z \neq B_z \partial_y$$

Some of the properties of a determinant in the definition of the cross product are therefore not automatically applicable to the definition of the curl as a determinant.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \langle C_x, C_y, C_z \rangle = \vec{i} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \vec{j} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \vec{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} =$$

(1.20) $$\vec{i}\left(A_y B_z - A_z B_y\right) - \vec{j}\left(A_x B_z - A_z B_x\right) + \vec{k}\left(A_x B_y - A_y B_x\right)$$

$$C_x = A_y B_z - A_z B_y; C_y = A_z B_x - A_x B_z; C_z = A_x B_y - A_y B_x$$

If we can factor out a certain scalar value from the vectors in the determinant, we can pull them to the front of the determinant. This is not correct if the vector **A** is a vector operator.

Assume that $\vec{A} = A \cdot \langle x, y, z \rangle$ and $\vec{\mathrm{B}} = 5\langle \mathrm{x}^2, xy, zx \rangle \Rightarrow$

(1.21) $$\vec{A} \times \vec{B} = 5\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A \cdot x & A \cdot y & A \cdot z \\ x^2 & xy & zx \end{vmatrix} = 5Ax\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ x & y & z \end{vmatrix} = \langle C_x, C_y, C_z \rangle$$

End of intermezzo on determinants.

Let us recapitulate the basic idea: If a path integral can be written as the integral of a perfect derivative, we can integrate it easily and the integral depends only on the initial and final point:

(1.22) $$\oint\limits_{\text{a to b along C}} \vec{F}(x, y, z) \cdot d\vec{r} = \oint\limits_{\text{a to b along C}} \frac{\partial S}{\partial x} dx + \frac{\partial S}{\partial y} dy + \frac{\partial S}{\partial z} dz =$$
$$\oint\limits_{\text{a to b along C}} \overrightarrow{grad} S \cdot d\vec{r} = \int_a^b dS = S(b) - S(a)$$

If the force involved in the calculation of the work is the gradient of a continuous scalar function, then its curl must be equal to 0. One can convince oneself of this fact easily by realizing that the curl of the gradient of a scalar function is always equal to 0:

$$\vec{\Delta} \times \vec{\Delta} S = \vec{0}$$

(1.23) $$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{\partial S}{\partial z} & \frac{\partial S}{\partial y} & \frac{\partial S}{\partial z} \end{vmatrix} = \vec{i}\left(\partial_y \frac{\partial S}{\partial z} - \partial_z \frac{\partial S}{\partial y}\right) - \vec{j}\left(\partial_x \frac{\partial S}{\partial z} - \partial_z \frac{\partial S}{\partial x}\right) + \vec{k}\left(\partial_x \frac{\partial S}{\partial y} - \partial_y \frac{\partial S}{\partial x}\right)$$

As you learn in calculus, if f is a continous function in x,y and z, then the mixed second derivatives are the same: $\partial_y \frac{\partial S}{\partial z} = \partial_z \frac{\partial S}{\partial y}$ etc.

Pay particular attention to the third term, which is the term of the curl of a function in the variables x and y only! It implies that a test for the path-independence of a force function in two variables consists in checking whether:

(1.24)
$$\vec{F}(x,y)=F_x(x,y)\vec{i}+F_y(x,y)\vec{j}$$
$$\oint_{\text{a to b along C}} \vec{F}(x,y)\cdot d\vec{r}=\oint_{\text{a to b along C}} F_x(x,y)\,dx+F_y(x,y)\,dy$$

If $\frac{\partial F_x(x,y)}{\partial y}=\frac{\partial F_y(x,y)}{\partial x}$ then the integral is path independent.

More generally, if the curl of the vector function under the path integral is 0, then the integral is path independent and the force can be written as the perfect derivative of a scalar function. In physics this scalar function S(x,y,z) is called the potential energy function U(x,y,z) of the force associated with it. The relationship is further defined as the negative gradient of U. Such a force is then called conservative:

(1.25)
$$\vec{F}=-\overrightarrow{grad}U=-\vec{\nabla}U$$

Here are some problems to get you acquainted with the cross product and vector operators

To practice the concepts of the vector operators a little bit more, consider for example, the gravitational force, which is a vector field (vector function with several variables). In this example the origin of the force is located at the center of the mass M. The force F is the force exerted by the mass M on the small mass m.

(1.26)
$$\vec{F}_g=\frac{-mMG}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\left(x\vec{i}+y\vec{j}+z\vec{k}\right)=-mMG\frac{\vec{u}_r}{r^2};$$
$$\vec{u}_r=\frac{\vec{r}}{r}=\frac{\left(x\vec{i}+y\vec{j}+z\vec{k}\right)}{\left(x^2+y^2+z^2\right)^{\frac{1}{2}}}$$

The x-component of the vector field $\vec{F}$ is

(1.27)
$$F_x=\frac{-mMG}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}x;\ \text{the y-component is } F_y=\frac{-mMG}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}y$$

Let us calculate the partial derivative of this function with respect to x: (This leads to the calculation of the curl of **F**.)

It is, using product rule and chain rule, which hold for partial derivatives just as for regular derivatives:

(1.28) $\frac{\partial F_x}{\partial x} = -mMG\left(\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}} - x\frac{3/2}{\left(x^2+y^2+z^2\right)^{5/2}}2x\right)$

Similarly for $\frac{\partial}{\partial y}\left(F_y\left(x,y,z\right)\right); \frac{\partial}{\partial z}\left(F_z\left(x,y,z\right)\right)$ and so on.

For example:

(1.29) $\partial_x F_y = \frac{\partial}{\partial x}\frac{-mMGy}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} = \frac{-mMGy}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}}\left(-\frac{3}{2}\right)2x$

Double-check that the curl of the force of gravity equals 0:

$$\vec{\nabla}\times\vec{F}_g \equiv curl\vec{F}_g = \vec{0}$$

One can show that **any force that is directed along the radial connection between the objects involved, and only depends on the distance r from a central point, is a conservative force. Such forces are called central forces.**

$$\boxed{\text{Central force: } \vec{F} = \pm f(r)\vec{u}_r = g(r)\left\langle x,y,z\right\rangle; g(r) = \frac{f(r)}{r}; r = \left(x^2+y^2+z^2\right)^{\frac{1}{2}}}$$

Prove that the curl of such a force is equal to zero!

We know that the relationship for every potential energy function U and the associated conservative force F is

vector function: $\vec{F} = \left\langle F_x, F_y, F_z\right\rangle$ each component is a function of x, y , and z

(1.30) scalar function: U=U(x,y,z)

$$\vec{F} = -\overrightarrow{grad}U = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) \equiv -\left(\partial_x U, \partial_y U, \partial_z U\right) = -\vec{\nabla}U$$

Assume that we know that the potential energy U of x, y, and z for the gravitational force is given by: (We will prove this later.)

(1.31) $U(r) = \frac{-mMG}{r} = \frac{-mMG}{\sqrt{x^2+y^2+z^2}}$; with the reference at ∞; $U(r\to\infty) = 0$

One can check that according to (1.30) $F_x = -\partial_x U$

The x component of the force of gravity $\vec{F}$ is given by:

(1.32) $F_x = -\frac{\partial}{\partial x}U(r) = -\frac{\partial}{\partial x}\left(\frac{-mMG}{\sqrt{x^2+y^2+z^2}}\right) = \frac{mMG\left(-\frac{1}{2}\right)2x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} = \frac{-mMG}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}x$

which is indeed the correct x component of $\mathbf{F}_g$

To repeat the general definition of work for a variable force on a variable path in several dimensions:

(1.33) $$\oint_{\substack{path\\ from\ \text{A to B}}} \vec{F} \bullet d\vec{r} = W = \oint_{\substack{path\\ from\ \text{A to B}}} F_x dx + F_y dy + F_z dz$$

The line integral above is equal to the work W done by the force **F** along a path, which must be defined with the path-integral. The result of this integral is in general path-dependent. If this integral is path-**independent** we call the associated force **F a conservative force**. There are several ways in which one can find out whether a force is path independent or not.

If one can find a function U(x,y,z), through inspection and/or guessing for example, such that

(1.34) $$F_x(x,y,z) = \frac{\partial U(x,y,z)}{\partial x}; F_y(x,y,z) = \frac{\partial U(x,y,z)}{\partial y}; F_z(x,y,z) = \frac{\partial U(x,y,z)}{\partial z}$$

then the force is conservative and does not depend on the path taken.

$$Example:$$

(1.35) $$\vec{F} = \langle ax, by \rangle; W = \oint_{\substack{\text{from } (x_A, y_A)\\ \text{to } (x_B, y_B)}} (axdx + bydy) = \frac{ax^2}{2}\bigg|_{x_A}^{x_B} + \frac{by^2}{2}\bigg|_{y_A}^{y_B}$$

Each term can easily be integrated separately, and our potential function in this case is clearly equal to: (1.36)

$$U(x,y) = \frac{ax^2}{2} + \frac{by^2}{2} = \frac{1}{2}\left(ax^2 + by^2 + \text{an arbitrary constant}\right)$$

Note that it does not matter for the previous statement if we add a constant to this function.

We say that: The potential energy function is determined up to a constant.

There are general methods to find the original function. You learn this in the class on differential equations. We limit ourselves here to the simple cases used in physics.

Summary:

Whenever we have to calculate the line integral ∫**F**·d**r** and the force function **F** derives from a potential function U (in physics we always have **F** = **-grad**U for any fundamental force), we have the simple situation that

(1.37) $$\boxed{\oint_{\substack{path\\ \text{from A to B}}} \vec{F}\cdot d\vec{r} = \oint_{\substack{path\\ \text{from A to B}}} \overrightarrow{-grad}U \cdot d\vec{r} = \oint_{\substack{path\\ \text{from A to B}}} -\vec{\nabla}U \cdot d\vec{r} = -\int_A^B dU = -\Delta U\Big|_A^B = W}$$

where A and B are the initial and final points of the integration path. **In the case of a conservative force the path does not matter, and the work depends only on the initial and final points.**

We have seen that there is a mathematical test to see if the function **F** is a conservative vector function. If the function **F**(x, y) has only two components, the partial derivatives must obey the following:

(1.38)
$$\oint F_x(x,y)dx + F_y(x,y)dy \text{ is path independent if:}$$
$$\frac{\partial F_x(x,y)}{\partial y} = \frac{\partial F_y(x,y)}{\partial x}$$

In the general case (force in three directions), we must have that the curl of F is 0.

(1.39)
$$a)\vec{\nabla}\times\vec{F} = curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix} = 0$$
$$\text{b)for example, let } \vec{\mathrm{F}} = \left\langle \mathrm{x}^2 yz, xz, xyz \right\rangle$$
$$c)C_x = \partial_y F_z - \partial_z F_y = xz - x; C_y = \partial_z F_x - \partial_x F_z = \mathrm{x}^2 y - yz;$$
$$C_z = \partial_x F_y - \partial_y F_x = z - x^2 z \text{ evidently not conservative!}$$
$$d)\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ \mathrm{x}^2 yz & xz & xyz \end{vmatrix} = (C_x, C_y, C_z) \text{ this is the curl in determinant form.}$$
$$\text{The third component is:} C_z = \partial_x F_y - \partial_y F_x \text{ For a two dimensional force in x and y}$$
this is the only component!

Let us calculate another simple example: Assume that **F** = <xy, 2x>.

(1.40)

(1.41)
$$\oint_{\text{a to b along C}} \vec{F}(x,y)\cdot d\vec{r} = \oint_{\text{a to b along C}} xydx + \cdot 2xdy$$
$$\partial_y(xy) = x \text{ and } \partial_x(2x) = 2$$

evidently this force function is path-dependent and not conservative. This means that you cannot find a function U for it, whose gradient would be **F**. If we calculate the work by this force along different paths we get different results.

Choose for example the paths (0, 0) along the x-axis to (2, 0), and then up the y-axis to the endpoint (2, 2).

Show that we get: for the first part of the path ∫(xydx + 2xdy) = 0, because y = 0

For the second portion upwards we get 2·2·y between 0 and 2, or 8. That is, along this path we get a total result equal to 8.

If we now choose a path along the straight line y = x, we get

∫(xydx + 2xdy) = ∫(x·x·dx + 2xdx) = $x^3/3 + x^2$ between 0 and 2, or 8/3 + 4 = 14/3

Now try the force function **F** = ($2x^2$, $3y^2$) and try to find the potential function by integrating. If we integrate a function in several variables, the integration of one variable is given up to a function of the other variables. (This function is equivalent to the integration constant when we integrate a function in one variable only.) If we can perform this integration we have found the potential function and the integral is path independent.

$$\int_{path} \vec{F}\cdot d\vec{r} = \int_{path} \langle 2x^2, 3y^2\rangle \cdot \langle dx, dy\rangle = \int_{path} \underbrace{2x^2dx + 3y^2dy}_{dU}$$

(1.42) $$\frac{\partial U}{\partial x} = 2x^2 \Rightarrow U(x,y) = \frac{2x^3}{3} + f(y); \frac{\partial U}{\partial y} = 3y^2 \Rightarrow U(x,y) = \frac{3y^3}{3} + g(x)$$

If we put the two results together we see that the functions f and g must be constants and get U(x,y)= $\frac{2x^3}{3} + y^3 + \text{constant}$

It is clearly path independent. Double check this result by calculating its gradient!

Let us state our results again:
(1.43)

> If the curl of a function **F** is 0, then its path integral is path independent. It derives from a scalar function U(x,y,z) which depends only on the coordinates. Such a function is called an irrotational vector field and is equal to –**grad**U. $\vec{\nabla}\times\vec{\nabla}U \equiv \vec{0}$

Kinetic energy and the law of conservation of mechanical energy:

Whenever the sum of the exterior forces is not equal to zero the mass on which the forces are acting will be accelerated according to Newton's second law:

$\sum_i \vec{F}_i = m\vec{a} = \frac{d\vec{p}}{dt} = m_0\frac{d\vec{\mathrm{v}}}{\mathrm{dt}}$ in the non-relativistic case. We obtain the work by applying the path integral to both sides of this equation:

(1.44) $$\mathrm{W}= \oint_{path\ A\to B} \sum_i \vec{F}_i \bullet d\vec{r} = \int_{path\ A\to B} m_0 \frac{d\vec{\mathrm{v}}}{\mathrm{dt}}\bullet d\vec{r} = \int_{path\ A\to B} m_0 d\vec{\mathrm{v}}\bullet\frac{d\vec{r}}{dt} = \int_{path\ v_A\to v_B} m_0\vec{\mathrm{v}}\bullet d\vec{\mathrm{v}}$$

(1.45)

After the variable transformation from $\vec{\mathrm{r}}$ to $\vec{\mathrm{v}}$, using $d\vec{\mathrm{r}}=\vec{\mathrm{v}}dt$,

$$\mathrm{W}= \int_{path\ v_A\to v_B} m_0\vec{\mathrm{v}}\bullet d\vec{\mathrm{v}} = m_0 \int_{path\ v_A\to v_B} \mathrm{v_x}d\mathrm{v_x} + \mathrm{v_y}d\mathrm{v_y} + \mathrm{v_z}d\mathrm{v_z}$$

this is a path integral but we can readily integrate it which means that it is path independent.

The reason for this can be seen easily:

(1.46)
Just like the function in the line integral $\int\limits_{path}$ xdx+ydy derives from $\frac{x^2+y^2}{2}$, so does the function $\langle v_x, v_y \rangle$ derive from $\frac{v_x^2+v_y^2}{2}$ and depends only on the initial and final velocity.

Therefore we get the final result that the work done by all exterior forces, including frictional, non-conservative forces is equal to the change in kinetic energy of our moving mass between the initial and final point.

(1.47)
$$W = \frac{1}{2} m_0 (\vec{v}_B{}^2 - \vec{v}_A{}^2) = \Delta K \text{ change in kinetic energy K.}$$
$$K = \frac{1}{2} m_0 \vec{v}^2$$

(1.48)
Work-energy theorem: The work done by all forces exterior to a mass m along a specific path is equal to the change of kinetic energy of the mass between the initial and final point of the path.

$$\oint\limits_{path\text{ A to B}} \sum_i \vec{F}_i d\vec{r} = W = \Delta K = K_B - K_A$$

(See also the calculation of the relativistic kinetic energy.)

Let us now assume that one of the forces in the path integral is a conservative force whereas the other is not:

(1.49)
$$\vec{F}_c = -gradU \text{ and } \oint\limits_{path} \vec{F}_c d\vec{r} = -\oint\limits_{\substack{path\text{ from}\\ \text{P1 to P2}}} \overrightarrow{grad} U \cdot d\vec{r} = -\int\limits_{P1}^{P2} dU = -\Delta U$$

$$W = \oint\limits_{path\ A\to B} \sum_i \vec{F}_i \cdot d\vec{r} = \int\limits_{path\ A\to B} (\vec{F}_c + \vec{F}_{\mathbf{nc}}) \cdot d\vec{r} = -\Delta U + W_{nc} = \Delta K$$

(1.50) *or*
$$W_{nc} = \Delta U + \Delta K$$

We again use the fact that the total derivative of a scalar field U(x,y,z) is equal to dU

(1.51)
$$dU = \overline{grad} U \cdot d\vec{r} = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz$$

If all forces are conservative, the left side is 0 and we have the **law of mechanical energy conservation:**

$$\Delta K + \Delta U = 0 \Rightarrow K_2 - K_1 + U_2 - U_1 = 0 \Leftrightarrow$$

$U_1 + K_1 = U_2 + K_2$ for any two points 1 and 2 during the motion of the particle with mass m.

(1.52) Another way of saying the same thing is that the total mechanical energy K+U of a particle is the same at all times. When the particle loses some kinetic energy it gains the same amount in potential energy, and vice versa.

When we consider macroscopic effects like friction, the work done by such dissipative forces is always negative and we can say that, see (1.50):

(1.53)

$$\boxed{W_{nc} = -f \bullet r = \Delta K + \Delta U}$$

The work done by friction forces is equal to the change in kinetic plus potential energy of the system. It is a negative value.

A rock of mass 500g is dropped by a distance of 100 m, and buries itself in the ground by a distance of 15.0 cm, calculate the average force of resistance over this distance of 15.0 cm. (3 significant figures)

Solution:

$$-f0.15 = 0 - 0 + 0 - mgh = 0.5 \cdot 9.8 \cdot 100.15$$

$$f = 327N$$

This work is usually dissipated as heat, which cannot be recovered to do some kind of work.
For example, when a bullet of mass 15 grams flying with a velocity of 550m/s, strikes a wall, most of the kinetic energy of the bullet is turned into heat energy Q (also sound and deformation energy).

If all forces are conservative we arrive at the important theorem for conservation of energy:
(1.54)

If all forces are conservative the total change of potential energy of all of these forces plus the total change in kinetic energy of the object on which these forces are acting is equal to 0. Or, the total energy of the system is conserved. (Hence the name conservative force.)

$$\Delta K + \Delta U = 0$$

$$K_2 - K_1 + U_2 - U_1 = 0$$

$$K_1 + U_1 = K_2 + U_2$$

End of regular stuff.

Advanced stuff (just for fun; omit it out if you have not yet had vector calculus):
An interesting path integral is the following which is equal to the area inside a closed curve, which is given parametrically:

(1.55) $$A = \frac{1}{2}\oint xdy - ydx$$

For example an ellipse has the parametric representation x=a·cos(t) and y= b·sin(t) where a and b are the major and minor axis, and t varies from 0 to 2π:

$$dx = -a \cdot \sin t \cdot dt; dy = b \cdot \cos t \cdot dt$$

$$A = \frac{1}{2}\oint a\cos t\left(b \cdot \cos t \cdot dt\right) - b \cdot \sin t\left(-a \cdot \sin t \cdot dt\right) =$$

$$= \frac{1}{2}\oint\left(ab\cos^2 t + ab \cdot \sin^2 t\right)dt = \frac{1}{2}\int_0^{2\pi} ab \cdot dt = \frac{1}{2}ab \cdot 2\pi = ab\pi$$

The proof of the formula requires the knowledge of Stokes' law:

(1.56) Stokes's law turns into Green's formula if we use just two dimensions. It relates a surface integral (closed surface without holes) to a line integral. We can use this to calculate the area of some geometric shapes given in parametric form, like the ellipse.

(1.57)

Stokes' theorem: The surface integral of the curl of $\vec{E}$ is equal to the line-integral along the line forming the boundary of the surface of the vector $\vec{E}$.

$$\iint_A \vec{\nabla} \times \vec{E} \cdot d\vec{\sigma} = \oint_{\partial A} \vec{E}d\vec{l}$$

If you let $\vec{E}$=(P, Q, R) be a vector function with an x and y component, namely P(x,y) and Q(x,y) the Stokes theorem turns into Green's formula

$$d\vec{\sigma} = \langle dydz, dxdz, dxdy \rangle$$

$$\vec{\nabla} \times \vec{E} = curl\vec{E} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & 0 \\ E_x & E_y & 0 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & 0 \\ P & Q & 0 \end{vmatrix} = \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\vec{\nabla} \times \vec{E} \cdot d\vec{\sigma} = \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \cdot dxdy\vec{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

Green's formula: $$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \oint_{\partial A} Pdx + Qdy$$

which for Q=x and P=-y becomes:

$$\iint_A 2dxdy = \oint_{\partial A} xdy - ydx \Rightarrow \text{area bounded by a curve:}$$

(1.58) $$\boxed{Area = \iint_A dxdy = \frac{1}{2}\oint_{\partial A} xdy - ydx}$$

One can derive an interesting relationship between angular velocity, velocity in circular motion, and the curl as follows. Calculate the curl of velocity in circular motions: (Don't ask if you must know this. What is there to know?). In circular motion the velocity vector is always tangential. Usually we write this velocity as $\vec{\mathrm{v}}=\omega r\vec{u}_\theta$

When we are dealing with a three-dimensional problem, like the rotation of a sphere, then the motion of a point along any circle on the sphere can better be described in terms of a cross-product, namely

(1.59)

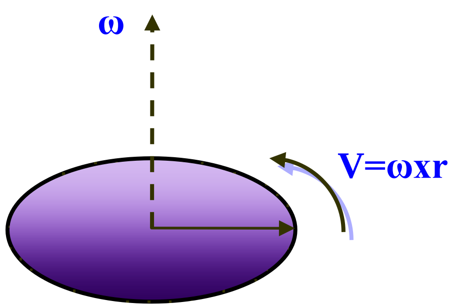
$$\vec{\mathrm{v}}=\vec{\omega}\times\vec{r}=\frac{\overrightarrow{d\theta}}{dt}\times\vec{r}$$

The angle dθ is the angle in circular plane of motion. The vector $\frac{\overrightarrow{d\theta}}{dt}=\vec{\omega}$ is perpendicular to the plane of motion.

$$\vec{\nabla}\times\vec{\mathrm{v}}=?\text{ with }\vec{\mathrm{v}}=\vec{\omega}\times\vec{r}$$

(1.60)

$$\vec{\omega}=\langle\omega_x,\omega_y,\omega_z\rangle;\ \vec{\mathrm{r}}=\langle x,y,z\rangle;\vec{\nabla}=\left\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\rangle;$$



(1.61)

$$\vec{\mathrm{v}}=\vec{\omega}\times\vec{r}=\langle z\omega_y-y\omega_z,x\omega_z-z\omega_x,y\omega_x-x\omega_y\rangle\Rightarrow$$

$$\vec{\nabla}\times\vec{\mathrm{v}}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ z\omega_y-y\omega_z&x\omega_z-z\omega_x&y\omega_x-x\omega_y\end{vmatrix}$$

$$2\langle\omega_x,\omega_y,\omega_z\rangle=2\vec{\omega}$$

To summarize:

(1.62)

The curl of velocity in circular motion equals 2ω:

$$\vec{\nabla}\times\vec{\mathrm{v}}=2\vec{\omega}\text{ or }\vec{\omega}=\frac{1}{2}\overrightarrow{curl}\vec{\mathrm{v}}$$

The circulation of the vector field $\vec{\mathrm{v}}$ can therefore be calculated:

(1.63)We apply a surface integral to both sides of the equation, and then use Stokes' theorem:

$$\vec{\nabla}\times\vec{v}=2\vec{\omega}\Rightarrow\underbrace{\iint_A \vec{\nabla}\times\vec{v}\bullet d\vec{A}}_{Stokes\ \text{law}}=\iint_A 2\vec{\omega}\bullet d\vec{A}$$

$$\oint_{\partial A}\vec{v}\cdot d\vec{s}=2\pi r v=2\omega\pi r^2\Rightarrow v=\omega r\Rightarrow\vec{v}=\vec{\omega}\times\vec{r}$$

When we study steady state liquid flow, we excluded the possibility of vortices. We can do this be requiring that the circulation of the velocity must be 0, which is the same as saying that the velocity vector can be written as the gradient of a scalar function.

Combinations of vector operators:

$a)\vec{\nabla}\vec{\nabla}=\vec{\nabla}^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\Delta$ called the Laplace operator, which can operate on a scalar function like in $\Delta f(x,y,z)$ [read: "Laplace f"] or on a vector function like in $\Delta\vec{E}(x,y,z)$.

We can also form $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right)$;

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right)=\vec{\nabla}\left(\vec{\nabla}\vec{B}\right)-\vec{B}\left(\vec{\nabla}\vec{\nabla}\right)=\vec{\nabla}\left(div\vec{B}\right)-\vec{\nabla}^2\vec{B}=grad\cdot div\vec{B}-\vec{\nabla}^2\vec{B}$$

$$div(f\vec{A})=fdiv\vec{A}+\vec{A}gradf$$

$$div(\vec{A}\times\vec{B})=\vec{\nabla}(\vec{A}\times\vec{B})=\vec{B}curl\vec{A}-\vec{A}curl\vec{B}=\vec{B}\vec{\nabla}\times\vec{A}-\vec{A}\vec{\nabla}\times\vec{B}$$

$$\vec{\nabla}\times f\vec{A}=curlf\vec{A}=gradf\times\vec{A}+fcurl\vec{A}=\vec{\nabla}f\times\vec{A}+f\vec{\nabla}\times\vec{A}$$

$$\Delta fU=f\Delta U+U\Delta f+2\vec{\nabla}f\cdot\vec{\nabla}U=f\Delta U+U\Delta f+2gradf\cdot gradU$$

$$u=\frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E}+\frac{1}{2\mu_0}\vec{B}\cdot\vec{B}$$

$\vec{S}=\frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)\ div\vec{S}=-\frac{\partial u}{\partial t}$ proof:

$\mu_0 div\vec{S}=\vec{\nabla}\left(\vec{E}\times\vec{B}\right)=\vec{\nabla}\left(\hat{\vec{E}}\times\vec{B}\right)+\vec{\nabla}\left(\vec{E}\times\hat{\vec{B}}\right)$ where we use the product rule for derivatives and $\hat{\vec{E}}$ means that ∇ operates on this vector only. We use cyclical permutation for

$$\vec{\nabla}\left(\hat{\vec{E}}\times\vec{B}\right)=\vec{B}\left(\vec{\nabla}\times\hat{\vec{E}}\right)=\vec{B}\left(-\frac{\partial\vec{B}}{\partial t}\right)=-\frac{1}{2}\frac{\partial\vec{B}^2}{\partial t}$$

$$\vec{\nabla}\left(\vec{E}\times\hat{\vec{B}}\right)=\hat{\vec{B}}\left(\vec{\nabla}\times\vec{E}\right)=\vec{E}\left(\hat{\vec{B}}\times\vec{\nabla}\right)=-\vec{E}\left(\vec{\nabla}\times\hat{\vec{B}}\right)=-\vec{E}\left(\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right)=\frac{-1}{c^2 2}\frac{\partial\vec{E}^2}{\partial t}$$

$$\mu_0 div\vec{S}=-\frac{\partial u}{\partial t}=-\frac{1}{2}\frac{\partial\vec{B}^2}{\partial t}-\frac{1}{2c^2}\frac{\partial\vec{E}^2}{\partial t}$$