Vector calculations

Vectors are ordered sequences of numbers. In three dimensions we write vectors in any of the following forms:

(1.1)
$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k} = \langle A_x, A_y, A_z \rangle$$

The quantities A_i are called the components of the vector $\vec{A}$;

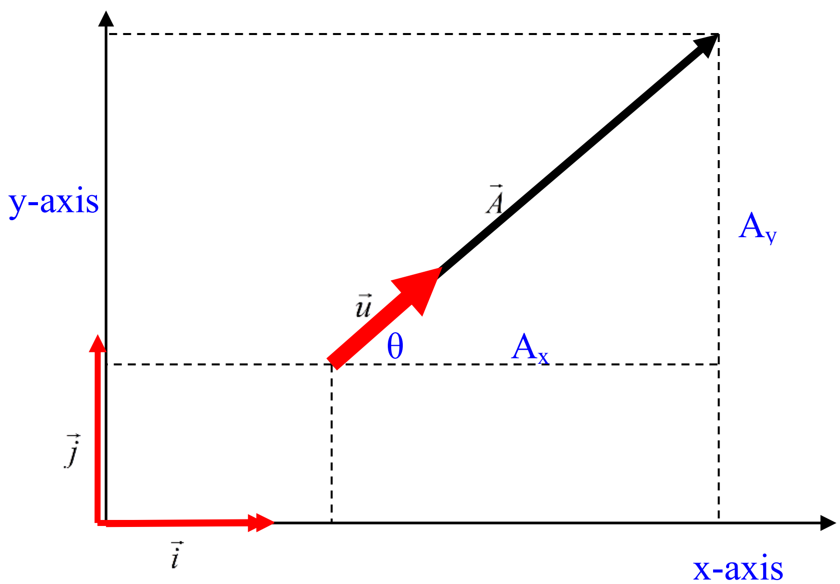
the magnitude of the vector $\vec{A}$ is called A (no arrow) and is equal to

$$A = |\vec{A}| = \text{ magnitude } A = \sqrt{A_x^2 + A_y^2 + A_z^2} \equiv \sqrt{\sum_{i=1}^{3} A_i^2} = \sqrt{\vec{A} \bullet \vec{A}}.$$

The dot indicates the scalar or dot product. The direction of the vector requires three angles in three dimensions, but fortunately only one angle in two dimensions.

In the case of two dimensional vectors we have:

$$\theta = \tan^{-1}\frac{A_y}{A_x}$$



Any location vector $\vec{r}$ can be written in terms of a directional unit-vector $\vec{u}_r$ and the magnitude of the vector which is r. A unit vector is any vector of **unit size**. We can obtain it by taking any regular vector and dividing it by its magnitude.

(1.2)
$$\vec{u}_A = \frac{\vec{A}}{A} = \frac{\vec{A}}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

We choose an orthogonal set of unit vectors, which together form the basis for our vector space. This means that any vector in this vector-space can be written in terms of a **linear combination of the unit-vectors**.

(1.3) $\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k} = \langle A_x, A_y, A_z \rangle$

The **Cartesian basis** consists of the fixed vectors $\vec{i}, \vec{j}, \vec{k}$ in the x, y, and z direction respectively. These are all unit vectors, which are perpendicular to each other. The components of a vector on this basis of unit vectors are the ***projections*** of the vector on the unit vectors.

Another set of unit vectors is used often in conjunction with circular motion. We use a **radial and a tangential unit vector** $\vec{u}_r$ and $\vec{u}_\theta$. We define the radial unit vector as the vector pointing **from the center of the circle to the point moving on the circle**. These unit vectors move with the particle. Their time-derivative is therefore not equal to 0.

So (1.4) $\vec{r} = r\vec{u}_r$ or for our vector $\vec{A}$ **= magnitude times unit vector**

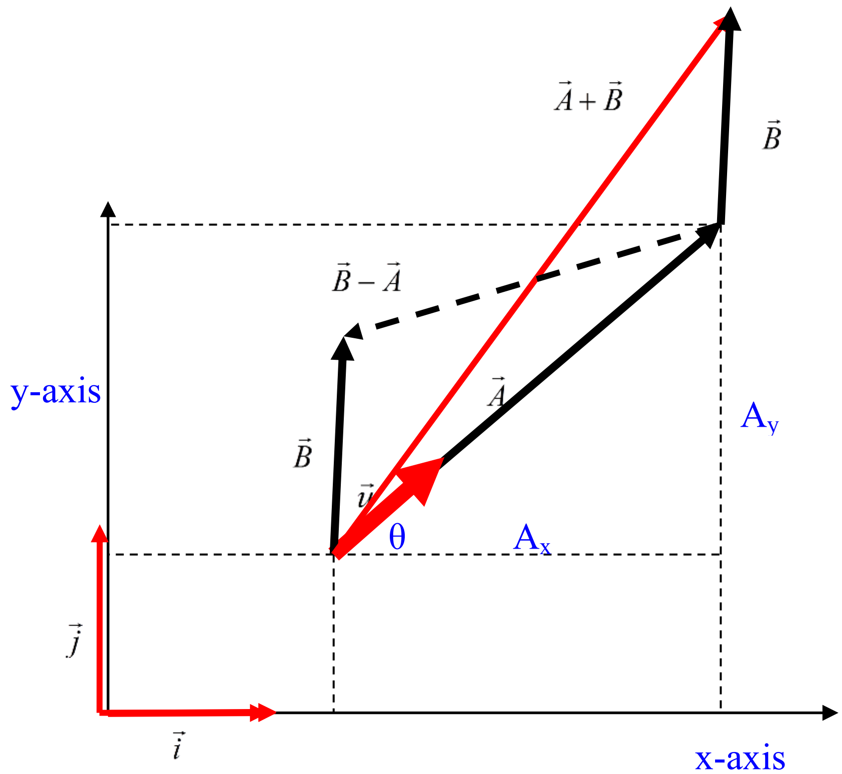
$$\vec{A} = A \cdot \vec{u}_A = \sqrt{A_x^{\,2} + A_y^{\,2}} \cdot \vec{u}_A \text{ with } \vec{u}_A = \frac{\vec{A}}{A} = \frac{\vec{A}}{\sqrt{A_x^{\,2} + A_y^{\,2}}}$$

If we add, subtract or multiply vectors we can do so using any coordinate system.

Addition of vectors:
(1.5)
$\vec{A} + \vec{B} = \langle A_x + B_x, A_y + B_y \rangle$ which corresponds to graphically forming a parallelogram out of the two vectors in such a way that the tail of one meets the head of the other. Head to tail, for short.

If we move the two vectors around parallel to themselves in such a way that their tails start at the same point, we create the difference between the two vectors. $\vec{B}$-$\vec{A}$ is the vector which starts at the end of $\vec{A}$ and goes to the end of $\vec{B}$.

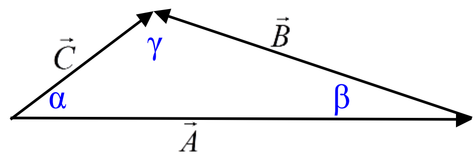
$\vec{B}-\vec{A} = \langle B_x - A_x, B_y - A_y \rangle$



Scalar Product:
(1.6)
$\vec{A}\cdot\vec{B} = A\cdot B\cdot\cos\theta = A_x\cdot B_x + A_y\cdot B_y + A_z\cdot B_z$ where the angle θ is the angle between the two vectors, which, by definition, is the smallest angle you get by putting the tails of the two vectors together. Measure angles always going counter clock wise, starting at the positive x-axis as 0. As you can see, the scalar product is just a number.

Now that we have defined the scalar product between two vectors we see that the components of a vector are the scalar products between the vector and the unit vector.
For example:
(1.7) $A_x = \vec{A}\bullet\vec{i} = A\cdot 1\cdot\cos\theta$: projection of the vector $\vec{A}$ on the x-direction.

We can apply the rule for scalar products conveniently to determine the law of cosines:



$$\vec{A}+\vec{B}=\vec{C}$$
$$\vec{A}-\vec{C}=-\vec{B}$$
$$\left(\vec{A}-\vec{C}\right)^2=\left(-\vec{B}\right)^2=$$
$$\vec{A}^2-2\vec{A}\bullet\vec{C}+\vec{C}^2=B^2=A^2-2A\bullet C\cos\alpha+C^2$$
$$\cos\alpha=\frac{B^2-A^2-C^2}{-2A\bullet C}$$

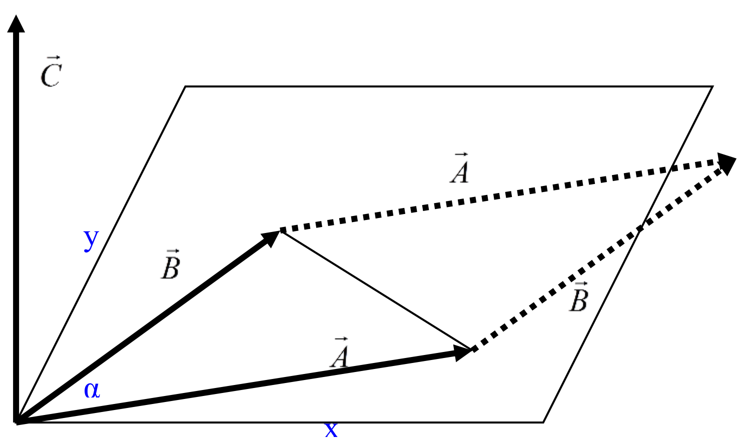
This is a nice proof of the law of cosines.
Find the angles for A=10, B=8, C=4

<u>**Vector Product or cross Product: (See also chapter 8)**</u>

Vector product:

$\vec{C}=\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$

The magnitude of the vector $\vec{C}$ is equal to $A\cdot B\cdot\sin\theta$ where θ is the angle between $\vec{A}$ and $\vec{B}$. The direction of the vector $\vec{C}$ is perpendicular to the plane spanned by $\vec{A}$ and $\vec{B}$, using the right hand rule. The thumb of your right hand points in the direction of $\vec{A}$, your index finger is $\vec{B}$, your middle finger, perpendicular to the other two fingers points in the direction of $\vec{C}$. (1.8)



The area spanned by the two vectors $\vec{A}$ and $\vec{B}$ is equal to AB sinα, which means that ½ AB sinα is equal to the triangle between the two vectors

(1.9)

$$\boxed{\begin{aligned}&\vec{C}=\vec{A}\times\vec{B}\\&\left|\vec{C}\right|=C=AB\sin\theta\end{aligned}}$$

One convenient way to calculate the cross product is by means of a **determinant**:

$$\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\left\langle C_x,C_y,C_z\right\rangle=\vec{i}\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}-\vec{j}\begin{vmatrix}A_x & A_z\\ B_x & B_z\end{vmatrix}+\vec{k}\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}=$$

$$\vec{i}\left(A_yB_z-A_zB_y\right)-\vec{j}\left(A_xB_z-A_zB_x\right)+\vec{k}\left(A_xB_y-A_yB_x\right)$$

(1.10) $$C_x=A_yB_z-A_zB_y;C_y=A_zB_x-A_xB_z;C_z=A_xB_y-A_yB_x$$

(1.11)

Assume that $\vec{A}=A\cdot\left\langle x,y,z\right\rangle=\left\langle A\cdot x,A\cdot y,A\cdot z\right\rangle\Rightarrow$ where A is a scalar.

$$\vec{A}\times\vec{B}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=A\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ x & y & z\\ B_x & B_y & B_z\end{vmatrix}=\left\langle C_x,C_y,C_z\right\rangle$$

Here is the rule: A common factor in a row or a column of a determinant can be pulled in front of the determinant.

Double vector product:

(1.12) $$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\cdot\left(\vec{A}\cdot\vec{C}\right)-\vec{C}\left(\vec{A}\cdot\vec{B}\right)$$

The double cross-product is another vector.

Mixed product: Cyclical permutation

$$\vec{\mathrm{A}}\cdot\left(\vec{\mathrm{B}}\times\vec{\mathrm{C}}\right)=\vec{\mathrm{C}}\cdot\left(\vec{\mathrm{A}}\times\vec{\mathrm{B}}\right)=\vec{\mathrm{B}}\cdot\left(\vec{\mathrm{C}}\times\vec{\mathrm{A}}\right)$$

(1.13) $$=\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$

The mixed product is a scalar, its absolute value is equal to the volume of the parallelepiped spanned by the three vectors.