Uncertainty (Error) Calculations In Scientific Measurements:

Significant Figures:

To be unambiguous in our notations of physical measurements we should write all numbers in terms of powers of ten. We should always do this in final results.

We also must always indicate the dimensions involved, for example we write:

1.7 mm = 1.7×10^{-3} m.

In both cases we use 2 significant figures. The number 0.0017 m also has just two significant figures, whereas the number 0.00170 m has three significant figures.
Whenever a number is given without indication of the ***relative*** or ***absolute error*** we assume that the error lies in the next significant figure and is equal to 5 units of that figure.

1.7 mm means therefore 1.7mm +/- 0.05mm.

When we multiply or divide numbers the term with the smallest number of significant figures prevails. We wait until the final result to round our numbers up or down.

x=1.334 y=3.45 z=2.6
$V=\pi \cdot x \cdot y \cdot z = 3.759223 \times 10^1$ cm^3 which we must round to 2 significant figures
or 3.8×10^1 cm^3

Let us say that we have measured the length of a table to be 3.654cm +/-0.05cm.
The number 0.005cm is called the absolute error or absolute uncertainty in the measurement of x. We write also Δx=0.05cm.

(1.1) $$\Delta x = 0.05cm$$ absolute error in x.

Δx is called the **absolute uncertainty** (error) in the measurement x, in contrast to the **relative uncertainty** $\frac{\Delta x}{x}$. **The relative uncertainty is a ratio of two numbers with the same dimension, therefore, the dimensions cancel out and the result is a fraction smaller than 1. We therefore often express the ratio as a percent ratio: 0.05 = 5%**

(1.2) $$\frac{\Delta x}{x}$$ relative uncertainty in x.

The value for x can also correspond to a function in several variables. Such a function could be f(x,y), for example:

$f(x,y) = 4x \cdot 5y^2$, where x = 5.34 cm +/- 0.004 cm and

y has been measured to be 33.68 cm +/- 0.008 cm. f = 1.21(147768)· 10^5 cm^3, where I have put the last meaningless six digits in parentheses.
To determine the error in this final result we would use 3 significant figures (sig. figs.), if we wouldn't know anything else about the uncertainties.

But we have in this case determined the absolute errors:
This means that Δx = 0.004 cm and Δy = 0.008cm. These are the absolute errors or uncertainties in x and y. Note that the absolute error in y is twice the absolute error in x, but that the relative error in y, Δy/y = 0.008/33.68 is much smaller than the relative error in x, Δx/x = 0.004/5.34.

If f is an arbitrary function in terms of powers of x, y, and z we can write:

(1.3)

$$f(x,y,z) = kx^a y^b z^c$$
$$\frac{\Delta f}{f} = |a|\frac{dx}{x} + |b|\frac{dy}{y} + |c|\frac{dz}{z}$$

For the example above that would give us:

(1.4) $$\frac{\Delta f}{f} = \frac{\Delta x}{x} + 2\frac{\Delta y}{y} = \frac{0.004}{5.34} + 2\frac{0.008}{33.68} = 1.22\cdot 10^{-3}$$

As the absolute error in the denominator is given in terms of 1 significant figure, the final relative uncertainty should also have only one significant figure or 0.1%.

Note that, even though the uncertainty itself is calculated with 1 significant figure, the result f itself must contain at least three significant figures.

To calculate the absolute error in f we multiply 0.00122 by f = 1.21147768· 10^5 cm^3

$$\Delta f = \frac{\Delta f}{f} f = 0.0012\cdot 1.21147768\cdot 10^5 cm^3 = 145.377...cm^3 = 2\cdot 10^2 cm^3$$

(1.5) the last number is rounded to 1 significant figure. So we must find the position for the $2\cdot 10^2 cm^3$, which is the fourth significant figure in f=$1.21147768\cdot 10^5 cm^3 \Rightarrow$

$$f = (1.212 \pm 0.002)\cdot 10^5 cm^3.$$

This means that our rule about significant figures is not accurate enough. In case of doubt, the calculated significant figures prevail over the simple rule about the smallest number of significant figures.

As absolute errors in a measurement are usually given in terms of 1 significant figure, the calculation of the relative and absolute uncertainties should also be rounded to 1 significant figure.

x=1.334cm+/- 0.0005cm; y=3.45cm+/- 0.005cm; z=2.6cm+/- 0.05cm

$$\frac{\Delta V}{V} = \frac{0.0005}{1.334} + \frac{0.005}{3.45} + \frac{0.05}{2.6} = 2.105486 \times 10^{-2} \Rightarrow 2 \times 10^{-2}$$

(1.6) $$\Delta V = \frac{\Delta V}{V} V = 2.105486 \times 10^{-2} \cdot 3.759223 \times 10^{1} cm^3 = 2.1 \times 10^{-2} \cdot 3.759 \times 10^{1} cm^3$$

$$= 0.79 cm^3 \Rightarrow 0.8 cm^3$$

We obtain the same result if we use the correct number of significant figures immediately:

(1.7) $$\Delta V = \frac{\Delta V}{V} V = 2 \times 10^{-2} \cdot 3.8 \times 10^{1} cm^3 = 0.8 cm^3$$

When measurements are ***added or subtracted*** we first express all numbers in the same dimension. Then we apply the rule that the **term with the smallest number of decimal places must prevail in the result**. (Note that here we are talking about decimal places, not significant figures.) The following terms in the sum have both 5 decimal places, the first number has also 5 significant figures, whereas the number 0.0004 has only one significant figure.

1.0001+0.0004=1.0005

1.04m+0.00007m=1.04m (three versus five decimal places)

$2.6\text{x}10^{3}$cm + $3.656\text{x}10^{-2}$km=$2.6\text{x}10^{1}$m+$3.656\text{x}10^{1}$m=6.7 $\text{x}10^{1}$m